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SOME NEW RATIOS OF CONIC CURVES.

By ALAN S. HAWKESWORTH.

Every conic curve is equidistant from a *fixed point* or focus [S or f], and a *fixed circle*, the *director circle*, whose center is the other, or second focus to the curve [S'], and whose radius [$S'D$] is equal to the major axis [AA' or aa', Fig. 1]. For, in the ellipse, $S'p+pf=aa'=S'D=S'p+pE$; while in the hyperbola, $S'P-PS=AA'=S'D=S'P-PE$. So that in both curves alike $pf=pE$, or $PS=PE$, for all possible positions of p or P.

In the parabola, since one focus [S'] is at infinity, the infinite director circle, whose center it is, must coincide with the usual rectilinear directrix to the curve. And since in every conic curve there are two precisely similar foci, either both real, or one real, and one ideal at infinity, either one of these can therefore be taken as the center of our director circle; the other focus thereby becoming our fixed point. But note: in the hyperbola, when that focus [S'], which lies within the opposite branch of the curve, is taken for our fixed point, and its director circle, as a consequence, lies partly or wholly within that branch which we are determining, then the equidistance of P must be measured, not along the shorter segment of the diameter, or produced diameter of the circle through P; but along its greater segment—*e. g.* in Fig. 1, QS is equal to QI, not to QK; and $P'S$ to PJ. And furthermore, when the distance SD of the fixed point from its director circle is twice, or greater than twice the radius of that circle—or in other words, when $S'S$ the inter-focal distance is thrice, or more than thrice AA' the major circle,—then the said circle cannot cut, but must lie wholly within

the opposite branch of the hyperbola. We may remark in passing that the term director circle, in place of ever meaning, as it undoubtedly should, this curvilinear directrix, is frequently misapplied to what is more properly called the *orthocycle;* which latter is in no sense a directrix; and merely happens to coincide with it, and the true director circle, in the special case of the parabola.

We can, still further, describe two unequal circles. The smaller, with its center at the fixed point, f or S, and with a radius which is less than half aa' or AA', the major axis. And the greater, with its center at S', concentric with the director circle; while its radius is less, or greater than that of said director circle, by the radius of the smaller circle, according as f or S was the point chosen for that smaller circle's center. In which case the ellipse or hyperbola, as the case may be, will evidently lie equidistant from the circumferences of the two unequal circles. So that any conic curve can also be generated by the equidistance between two such circles, which, in the special case of the parabola, become a circle and a right line.

Returning to the director circle and its fixed point, we may trace the sequence of the generated curves as follows. Commencing with the circle, wherein the fixed point and the center of its director circle are one, we pas immediately into the ellipse, as soon as said fixed point and center separate. The ellipse becoming flatter and flatter as the fixed point moves towards the circumference of its circle. Nor does the direction of this approach make any difference; since each fixed point, or focus, must have, on the opposite side of the bifocal ellipse, its precisely similar twin focus, and center of its director circle. Flatter and narrower grows the ellipse, until the point having approached infinitesimally close to the circumference—or, what amounts to the same thing, the center of the director circle having become, proportionately, infinitely distant from its fixed point—our ellipse passes, momentarily, into a pair of parabolas. But the instant that the "becoming" of the infinitesimal approach ceases, and the point is now actually *on* the circumference, then these momentary parabolas vanish; and the two sides of our degenerated conic coalesce with its major axis into one horizontal right line, running from said fixed point on the circumference in either direction. Said circumference of the director circle being now represented by a perpendicular right line through the point; since it is, proportionately, infinite in radius. The infinitesimal progress reappearing, as our point crosses the circumference [now, once more, of finite radius] and recedes infinitesimally away from it, this degenerated line [extended, now, externally along the major axis] splits; first, for an instant, again into a second pair of parabolas, and then into a narrow hyperbola. Said hyperbola becoming broader and broader, and its two opposite branches, proportionately, closer and closer, as the fixed point recedes from its director circle. Until point and circle having become infinitely distant, and therefore the radius $S'D$, which is equal to the major axis AA , proportionately infinitesimal, the two opposite branches of our hyperbola, fusing into one, have degenerated once more into a right line, but now lying at right angles to the major axis. And similarly, while our point had as yet moved but

an infinitesimal distance from the circumference, then the conjugate curve to the resultant extremely narrow hyperbola, or pair of parabolas, being extremely broad, must have a major axis and director circle of infinitesimal magnitude; strictly corresponding, then, in its resultant ratios to one immeasurable distant. While as the fixed point of the original curve recedes from its circle, so reciprocally do the director circles of its conjugate curve grow in proportionate magnitude; and the comparative distance of their respective points decrease. Until, said fixed point of the parent curve having passed to infinity, and its hyperbola therefore, as already stated, degenerated into a right line, perpendicular to the major axis, the conjugate curve must now possess director circles infinite in radius, and infinitesimally close to their respective fixed points; and thus be, first, momentarily, a pair of parabolas, and then a similar pair of degenerated right lines, lying parallel to, and indeed inside of, and coalescing with, the degenerated original curve. At the one extreme, then, of this sequence that we have traced, lies the circle, with its foci coalesced into the central pole. And at the opposite extreme, as the farthest possible removed form, lies the degenerated "perpendicular right line hyperbola," with its infinitely distant fixed point on the polar of said pole. While midway lie the two sets of parabolas, facing in opposite directions, and each flashing momentarily into being, as the ellipse sinks into, or the hyperbola arises from their medial point of indifference; which is, again, a degenerated horizontal right line.

From the above the following theorems can be deduced:

Theorem 1. Even as the sum, in the ellipse, or difference, in the hyperbola, of the two focal distances of any point P on the curve is constant, and equal to the major axis; so, similarly, is the sum or difference of its distances from the two director circles.

Theorem 2. The line SE or fE, SH or fH, etc., joining the fixed point S or f to the extremity of any radius $S'E$ or $S'H$ of its director circle, is ever bisected by the auxiliary circle to the curve. While if the two radii, in the ellipse, or diameters in the hyperbola, of the two director circles be drawn through the same point P or p on the curve, and their extremities joined with their respective fixed points [S or f, as the case may be]; and the bisections of such two lines, by the auxiliary circle, be joined by a right line, then will this line be the tangent at P or p.

Theorem 3. In the ellipse, the produced latus rectum, which is the double sine through f, the fixed point, will intersect upon the director circle those two radii, which determine the minor axis of the curve. For if $S'B=Bf=BH$ [Figure 1], then $S'fH$ are concyclic; $S'fH$ a right angle; and thus fH the produced semi-latus rectum.

Theorem 4. While correspondingly, in the hyperbola, those two diameters of the director circle, whose respective tangents pass through the fixed point, are parallel to the asymptotes, determining at infinity the ideal minor axis of the curve. Which two tangents, furthermore, are perpendicularly bisected by those asymptotes, at their common intersection with the rectilinear directrix.

For, let $S'H$ be the radius, whose tangent HS passes through S, the fixed point [Figure 1]. And let ZCZ''', $Z''CZ'$ be the asymptotes, meeting the directrix in YY'. Then $S'H:S'S=AA':SS'=CA:CS=CY:CS$. So that the right-angled triangles $S'HS$ and CYS are similar, with $S'H$ parallel to the asymptote CYZ; and SY half of HS, even as CS is of $S'S$. And thus CYZ bisects SH at right angles.

Corollary 1. If a perpendicular Cb were raised at C the center of the curve, and through either of the apeci A or A' a line Ab or $A'b$ be drawn, parallel to either $S'H$, or an asymptote, it will cut Cb in b, an extremity of the accepted minor axis of the hyperbola—the major axis of its conjugate.

Theorem 5. If from any point, say K, upon the circumference of the director circle, two tangents KI and KG be drawn to the curve, cutting the circle again in GI, and these points GI be joined; then will said GI be a third tangent to the curve. And the resultant circumscribing triangle KGI will have for its orthocenter S [or f, as the case may be], the fixed point.

Taking the hyperbola [Figure 1]. There are evidently innumerable circumscribing triangles possible. And similarly, innumerable triangles, having S, the fixed point, for their orthocenter. And finally, innumerable triangles which fulfill both of these conditions.

Let one of these latter, KGI, be drawn, both circumscribing the hyperbola, and with its orthocenter at S. We will prove that its vertices KGI lie on the director circle.

The auxiliary circle to the curve cuts the three tangents KI, KG, and GI in points vi, wk, and ug, respectively. Then, by a well known theorem, SiI, SkK, and SgI are right angles; and thus gki are the pedal points of the perpendiculars from the vertices upon the sides of the triangle KGI. And therefore the auxiliary circle, as the circumscribing circle to this pedal triangle kgi, is the "nine points circle" of triangle KGI; having its center C collinear with, and bisecting the distance between the orthocenter S and circumcenter S' of KGI; while its radius is half that of the circumscribing circle, since it is itself the circumscribing circle to the medial triangle uvw. So that since $CS=CS'$, and $CA=\frac{1}{2}AA'=\frac{1}{2}S'D$, therefore the director circle is the circumscribing circle to the triangle KGI, tangential to, and thus circumscribing the hyperbola, and with its orthocenter at the fixed point S.

An additional proof is as follows. If uvw be the points where the sides of the circumscribing triangle KGI are cut by its "nine points circle"—the auxiliary circle,—then such points thereby bisect those sides; and the perpendiculars $S'u$, $S'v$, and $S'w$ upon them must thus meet in S', the circumcenter to the triangle, and second focus to the curve.

And since the same proof holds true, and the auxiliary circle is the "nine points circle" for any triangle, which is both tangential to the hyperbola, and has its orthocenter at S; then, conversely, any two tangents from any point upon the circumference of the director circle must determine a third tangent to the curve; and the resultant circumscribing triangle have its orthocenter at S, the fixed point.

In a like manner, if a triangle be drawn circumscribing an ellipse, and with its orthocenter at f, the fixed point of the curve; then the auxiliary circle is still the "nine points circle," and thus the director circle its circumscribing circle.

In the parabola, since the director circle, being infinite in radius, has become the rectilinear directrix, no real circumscribing triangle, having its orthocenter at the focus S, can be drawn. Although the tangent on the vertex, which represents, of course, the auxiliary circle, would also be the infinite "nine points circle" to any such triangle; since it passes through the pedal points of the perpendiculars from the orthocenter S upon the tangents.

Corollary 1. The auxiliary circle, as the "nine points circle," bisects not only the sides KG, KI, and GI of the circumscribing triangle; but also the three lines SK, SG, and SI, joining its vertices to its orthocenter. While, if the perpendiculars Sg, Sk, and Si be produced, they will ever meet their circumcircle, the director circle, at double the distances Sg, Sk, and Si, respectively.

Corollary 2. The "nine points circle" is thus tangential, not only to the inscribed and escribed circles of triangle KGI, but also to its inscribed ellipse, or escribed hyperbola.

Corollary 3. If the pedal triangle kgi be drawn, then, by known theorems, its sides must be equally inclined to, and thus its angles be internally or externally bisected by the sides of its parent triangle KGI; whose three apeci K, G, and I, and orthocenter S, furthermore, must ever be the centers of its inscribed and escribed circles. While points SkI, SGi and SgK, like KGk, must be ever collinear, in ellipse, or hyperbola. So that points SGI, SGK, and SKI must ever lie on the arcs of circles, whose radii are equal to each other, and to that of the director circle—namely, to the major axis.

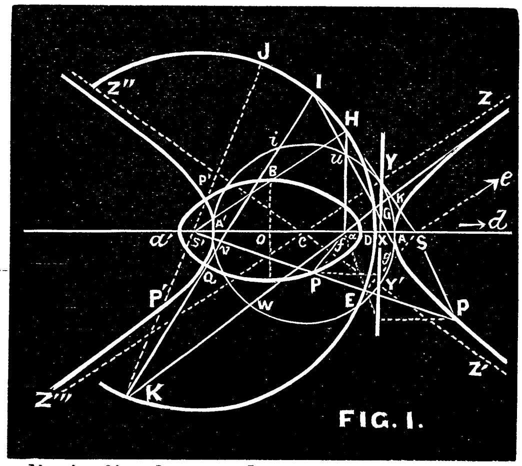

FIG. I.

But the most important deductions from the foregoing are the following theorems; by means of which the exact opposite and reciprocal reverse, point for point, of any conic curve can be precisely determined. A parabola reversing into an equal, but opposite parabola; a circle into a perpendicular right line; and an ellipse into an exactly reciprocal hyperbola; and conversely; with equal major axi.

Theorem 6. If an ellipse and a hyperbola, with major axi lying collinearly along one right line, and with a common director circle [and thus also with equal, though not coincident, major axi], have their respective fixed points each on the polar of the other, in respect to their common circle, *i. e.*, [Figure 1] $S'f{:}S'D = S'D{:}S'S$; then said ellipse and hyperbola will be each the reverse or

reciprocal of the other. The common radii of their common director circle determining corresponding points upon each of the two curves, with focal distances which are equally inclined to their respective major axi.

Thus, if $S'pEP$ be a common radius of the common director circle, cutting and determining the ellipse in p, and the hyperbola in P, then will angles pfS', and PSd, or pfa and PSA be equal. Similarly, the same common radius will determine the lati recti in both curves. Or finally, that radius $S'H$, lying parallel to an asymptote of the hyperbola [Theorem 4], thus determining at infinity its ideal minor axis, determines also the minor axis of the reciprocal ellipse [Theorem 3]; and subtends an equal angle in both curves; BfS' of the one equalling eSd of the other. And so on, for all the radii.

Taking first the radius $S'H$ parallel to the asymptote ZCZ''. Then by construction, $S'f{:}S'H=S'H{:}S'S$; so that the triangles $S'HS$ and $S'fH$ are similar right angled triangles; and fH is thus the produced semi-latus rectum of the ellipse. Therefore $S'H$ must pass through and determine B, an extremity of the minor axis of the ellipse [Theorem 3]; even as it also meets the asymptote of the hyperbola at infinity. While by construction, angles $HS'f$, ZCd, and eSd are equal, and thus also BfS' and eSd.

Next, taking any other radius $S'pEP$; determining the ellipse in p, and the hyperbola in P. Join pf, PS, Ef, and ES. Then again $S'f{:}S'E=S'E{:}S'S$; and so $S'ES$ and $S'fE$ are similar triangles, with equal angles opposite to homologous sides, $S'ES$ to $S'fE$, and $S'Ef$ to $S'SE$. But pE being equal to pf, and PE to PS, angles pEf and pfE are equal; and also PES to PSE. And therefore angles $pES-pEf$, or fES equal $S'fE-pfE$, or pfS'. While on the other hand, angles $S'SE+PSE$, or PSS' equal $S'Ef+PES$. So that their supplementary angles PSd and fES are equal; and therefore $pfS'=PSd$, and $pfa=PSA$.

And similarly, for any radius of the common director circle. So that if the one be taken which determines fl, the semi-latus rectum in the ellipse, and cuts the hyperbola, say, in L; then angle LSd can be shown equal to the right angle lfS'; and thus LS be the semi-latus rectum of the hyperbola.

Said ellipse and hyperbola are thus reciprocal curvilinear forms; or the exact reversals and opposites of each other.

But note: when a radius beyond $S'H$ is taken, as *e. g.* $KS'J$, determining therefore upon its *diameter* the reciprocal points p' on the ellipse, and P' on the other branch of the hyperbola, then the axial angle of P', which is equal to $p'fS'$ and fJS is, not $P'Sd$, but $P'SS'$. So that the axial angle, in the hyperbola, must thus in every case be measured towards its ideal minor axis, as it is towards the real, in the ellipse.

If in place of f and S, we choose S' for our focal angle, then our theorem still holds true; for obviously, $PS'd=pS'f$, and $P'S'A'=p'S'a'$, etc.

Applying this theorem to the parabola, since it can be considered either an extreme ellipse, or an extreme hyperbola, our reciprocal curves will take the form of two precisely similar parabolas, facing in opposite directions a common directrix. In which case, manifestly the above theorem holds true. While if

we take a circle as our ellipse: then since $S'f$ its inter-focal magnitude equals zero, its polar to f must lie at infinity. And thus its reciprocal hyperbola, or reverse curvilinear form, whose fixed point S is on that polar, must assume, as already pointed out, the degenerate shape of a right line, lying at infinity, perpendicular to the collinear major axi.

If, then, we draw about point S' two concentric circles; the larger with double the radius of the smaller, and representing its director circle, and let them be cut in p and E respectively by a radius $S'pE$. Then $S'pE$ produced will represent the direction in which, at infinity, lies the ideal point P upon this degenerated right line hyperbola. While any other radius $S'aD$, making less than a right angle with $S'pE$, may represent the collinear major axi; and thus a line E..........S, through E, parallel to $S'aD$, will represent the direction at infinity of the ideal fixed point S of this degenerated hyperbola. Then, by hypothesis, $PE=PS$; and thus the ideal angles PSE, PSa, or PSA equal the real angles PES and $pS'a$. So that our theorem is again true.

Corollary 1. Thus if any points p and P upon the reciprocal curves subtend equal angles pfS' and PSd, or pfa and PSA; then they are thereby corresponding points, lying on one common determining radius or diameter.

Corollary 2. If pm and PM, their perpendiculars to the directrix, be drawn, then angles fpm and SPM are equal.

Corollary 3. If the lines fE and SE, or fJ and SJ, joining the respective fixed points to the extremity E, or J, of the common determining radius $S'E$, or diameter $P'S'p'J$, be each bisected in r and R by the respective auxiliary circles [Theorem 2], then the line rR, joining said medial points, is thereby fixed, both in direction, and magnitude; being parallel to, and one half of, fS.

Theorem 7. The said reciprocal curves, ellipse and hyperbola, having a common director circle, have thereby also both reciprocal eccentric ratios $S'f:aa'$, or $fa:aX$ of the one, equaling $AA':S'S$, or $AX:SA$ of the other; and a common rectilinear directrix YXY'. Which, furthermore, bisects the reciprocal focal distance, *i. e.*, fS. So that $fa:aX$ of the ellipse is not only of equal ratio to its reciprocal $AX:SA$ of the hyperbola; but is also of equal magnitude, fa equalling AX, and aX equalling SA [Figure 1].

For, bisecting fS in X by the perpendicular YXY', then $fD:DS=fD:fS-fD=fS-DS:DS$, and thus their halves are in a similar proportion; $fa:fX-fa=XS-AS:AS$; or $fa:aX=AX:SA$. While $fa+aX=XA+AS$; and so $fa=AX$, and $aX=SA$.

Lastly, if O be the middle point of $S'f$ in the ellipse, and C of $S'S$ in the hyperbola, then will $fa:aX=fD:DS=S'f:S'D=S'f:aa'=Of:Oa=Of+fa:Oa+aX=Oa:OX$. So that X is on the polar of f in the ellipse; and thus the polar YXY' is the rectilinear directrix.

And similarly, in the hyperbola, $SA:AX=SD:Df=S'S:S'D=S'S=A'A=CS:CA=CS-SA:CA-AX=CA:CX$. So that YXY', as the polar of S, is the rectilinear directrix of the hyperbola.

To draw reciprocal conic curves, then, we may either, with a common

director circle, place their respective fixed points each on the polar of the other; or else, using a common rectilinear directrix, we may make their eccentric ratios reciprocally equal, both in proportion, and in size. The said eccentric ratio $fa:aX=AX:SA$, or polar proportion $fD:DS=S'f:S'D=S'D:S'S$ being the same in both cases. While in either case, also, the major axi are both collinear, and equal; and the second focus S' is in common.

Corollary 1. If the focal distance pf of a point p on one of these curves equals in magnitude the distance PM from the directrix of point P on its reciprocal curve, then conversely, must the directrix distance pm of the first point p equal the focal distance PS of the second, P.

Corollary 2. Since $SY=YH$ [Theorem 4], and $S'B=Bf=BH$, therefore $YX=OB$. And thus YY', the portion of the directrix cut off between the asymptotes of the hyperbola, ever equals the minor axis of the reciprocal ellipse.

Corollary 3. fX or SX, being the half of fS, equals the distance OC between the respective centers of the curves. For $S'S-S'f=fS$; and thus $\frac{1}{2}S'S-\frac{1}{2}S'f=\frac{1}{2}fS=fX=SX=S'C-S'O=OC$. And this distance, again, must equal the space between the several apeci, Aa, or $A'a'$. Since $fa=AX$, and $aX=SA$, and thus $fa+aX=fX=AX+aX=Aa=fa+SA=S'a'+S'A'=A'a'$.

Theorem 8. If D be the point where the director circle cuts the major axis of a conic; $S'E$ or $S'J$ any radius or diameter of this circle, determining a point P or p on the curve; DE a line joining points D and E; and ES or Ef a line from E to the fixed point S or f. Then will angle DES or DEf, as the case may be, ever be one half of the axial angle of P or p; which, in the ellipse is angle pfS', but in the hyperbola PSd, or PSS' [Theorem 6], according to the branch which point P may be upon [Figure 1].

For in Theorem 6 angles fES, pfS' and PSd were shown to be equal; while $S'fE$ and $S'ES$ being similar triangles, the homologous sides $fE:ES=S'f:S'E=S'E:S'S=S'D-S'f:S'S-S'D=fD:DS$. So that DE must bisect angle fES [Euclid VI, 3]; and thus $DEf=\frac{1}{2}pfS'$; and $DES=\frac{1}{2}PSd$.

Were $P'S'p'J$ the determining diameter chosen, then in a like manner it can be shown that $fJ:JS=fD:DS$; and thus angles $DJf=DJS=\frac{1}{2}fJS=\frac{1}{2}p'fS'=\frac{1}{2}P'SS'$.

In the parabola, PM, the perpendicular through P to the directrix, obviously represents pE or PE as above; and since angles XSM, SMP, and PSM are all equal, SM must bisect angle XSP; and thus SMX, the complement to SMP, be half of PSd, the supplement to XSP.

While, in the circle, making the same construction as in Theorem 6, angles $S'ED$, $S'DE$, and DES are all equal; and thus DES' or DES is one half of the supplement to $p'S'a$, or to the ideal angle PSa.

NON-EUCLIDEAN SPHERICS.

By DR. GEORGE BRUCE HALSTED.

As part of the general enlightenment springing from the creation of non-Euclidean geometry, every one now knows that spherical trigonometry is entirely independent of the parallel postulate. We wonder that anyone should have stopped to give proof of what is now so obvious.

Yet perhaps the most difficult article in all Lobachevski's "Geometrical Researches on the Theory of Parallels" is §35, which concludes, "Hence spherical trigonometry is not dependent upon whether in a rectilinear triangle the sum of the three angles is equal to two right angles or not."

Just so concludes Chapter XI of his New Elements: "Therefore the equations for spherical triangles remain the same whether we assume the angle of parallelism as constant or variable."

In §26 of Bolyai's Science Absolute of Space spherical trigonometry is established independently of the parallel postulate.

In his non-Euclidean space Bolyai found a uniform surface, F, whose proper geometry is Euclidean, its straight being L, the circle-limit. Lobachevski found the same, calling F orisphere, L oricycle.

But profound as was their genius it never questioned the assumption, of every three costraight points always one and only one is between the other two. As a consequence, the straight was for them of essence unclosed, and space infinite. So the characteristic geometry of the sphere was not given rank with that of the orisphere, and the non-Euclidean geometry of finite space remained unsuspected. Neither reached the conception that the totality of space may be finite.

The circle-limit was infinite and was conceived as the straight of the orisphere; the finite great circle was not conceived as the straight of the sphere. It remained for Riemann to perceive that the straight, though unbounded, need not be infinite, whence followed a new non-Euclidean geometry, now called by his name.

Beltrami showed that in Euclidean space there may be a surface a piece of which may perhaps represent a piece of the Bolyaian plane. Such is the surface of constant negative curvature, the pseudosphere.

True, it is impossible to represent the entire Bolyaian plane on a Beltrami surface without singular points; nevertheless, meaning by pseudospheres surfaces of revolution which have for meridians a tractrix or curve of equal tangents, we may say their characteristic geometry is Bolyaian.

Of late this doctrine has been filled out, completed by the beautiful

THEOREM OF BARBARIN: *Each of the three spaces, Euclidean, Bolyaian, Riemannean, contains surfaces of constant curvature of which the geodesic lines have the metric properties of the straights of the three spaces.*

These surfaces are (1) the tubes or surfaces equidistant from a straight,

it being possible for the distance to be infinite, which gives the orispheres, (characteristic geometry, Euclidean); (2) the pseudospheres (characteristic geometry, Bolyaian); (3) the spheres (characteristic geometry, Riemannean).

In 1879 Killing made clear the distinction between Riemannean space and what he then called its polar form, by Klein called simple elliptic space. This latter, Killing thinks, had been entirely missed by Riemann, as we know it was by Helmholtz even up to 1876 when he still reproduced the old but false theorem that in space of positive curvature two geodetic lines, if they in general intersect, must necessarily intersect in two points. Such a space of constant curvature Klein calls *spherical* in contradistinction to the simple elliptic in which the assumption, two points determine a straight, has no exception.

Killing it was also who first proved that besides the Euclidean, Bolyaian, simple elliptic, the spherical or old Riemannean is the only, the sole space which as a whole can be freely moved in itself. There are undertypes in abundance where the free mobility of figures only holds so long as the dimensions of the figures do not surpass a certain size; a series of topologically distinguishable spaces which for bounded (simply connected) parts are Euclidean, Bolyaian, simple elliptic. Moreover it has been demonstrated, so far as concerns the surfaces of constant positive curvature, to which the Riemannean geometry applies, that apart from the sphere there is no other closed surface of this sort. The sphere is the only closed surface of positive constant curvature without singularities.

All this intensifies the importance of surface spherics, two-dimensional spherics, pure spherics, intrinsic spherics, Riemannean spherics, double-elliptic spherics, non-Euclidean spherics.

Fortunately a place even in general education has been held open for this newcomer. All the theorems of the so-called "solid geometry" of the schools which relate solely to the surface of the sphere, there obtained by using the parallel postulate, by dragging in the globe which in Euclidean geometry is inside the sphere, in fact by considering the sphere as the covering belonging to such a globe, and therefore tri-dimensional, in a three-way-infinite manifold, are really theorems of that simpler finite manifold the sphere, having no dependent relation to the Euclidean straight, plane, or space. How obvious, then, that these theorems should be developed entirely from the assumptions which characterize the sphere.

Even what is ordinarily conceived of as the shape of the sphere is not wholly irrelevant, for, using the terminology of our Euclidean intuition, if the surface or covering of a globe be detached from the globe, any surface into which it can be bent without stretching, into which it can be flexed, is a double elliptic surface, a surface of constant positive curvature with its proper geometry, and what is ordinarily thought of as the free mobility of figures in it may remain; but somewhere on this surface has come a singularity, it is no longer free from singularities.

As an illustration in still lower terms of the meaning of intrinsic properties of the sphere, take the circle, the closed curve which will slide on its trace,

which is mobile in itself as a whole. This occurs in the Euclidean, Bolyaian, Riemannean plane with its intrinsic properties unchanged; but consider its radius, and it flies apart into three. The circumference of a circle in the Euclidean plane equals $2\pi r$; in the sphere the circumference is less than $2\pi r$; in the pseudo-sphere the circumference is greater than $2\pi r$. The intrinsic properties of the sphere are just what we want. Since they are utterly independent of the parallel-postulate, the simplest spherics must be non-Euclidean.

The student's familiarity with the sphere under its old Euclidean aspect as glove to a globe is also an advantage, which to all needing an introduction to the new ways of treating geometries decides in favor of intrinsic spherics as against simple elliptic planimetry with its unilateral plane which we can so strangely get through without going through. In spherics we have familiar material to present in the new light, with the new methods, to be therein acquired that they may be then retained for analogous conquest of unfamiliar realms.

When instead of building up theorems on polyhedral angles and then cutting them back into spherics, we realize the more complex theorems of angloids as already given in the simpler points of the sphere, we appreciate the practical in the theoretic.

How important, how enlightening to set forth the fundamental assumptions which give by pure logic all the relations of spheric figures, and to see developed therefrom the familiar system of theorems which so long constituted spherical geometry.

The old straight line, the old great circle are dissipated, volatilized, and in their place comes the *straightest* to which now applies the old definition of the straight line, "a line which pierces space evenly, so that a piece of space from along one side of it will fit any side of any other portion."

In vulgar phrase, the straightest turns neither to the right nor to the left so far as the sphere is concerned. But motion can never be fundamental, and it is the assumptions which really make the space. There is one geometric entity back even of the straightest, the point. It is the relation of the straightest to the point which differentiates the spheric from the simple elliptic. "The straight," says Mansion, "is a line determined by any two of its points, sufficiently near." But what is the meaning of 'sufficiently near'?

As long ago as 1877, I overcame these difficulties by building up the system of spherical geometry on a set of assumptions expressing only the few fundamental relations of points and straightests. Clearness is subserved by using 'straightest' as the designation for the spheric straight.

Line is a word which has always been used for the genus of which curve is a species, and of late such distractingly, bewilderingly complex curves and lines have appeared, that line in general should have no longer a place in the elements. Point and straightest are consciously accepted as elements to which specific assumptions give the requisite precision. To forestall controversy, one may reserve the word definition to mean an agreement to substitute a simple term or symbol for more complex terms or symbols.

Instead then of Mansion's definition we have what we prefer to call an Assumption of Association: I 1. For every point of the sphere there is always one and only one other point which with the first does not determine a straightest.

This second point we will call the *opposite* of the first. After three more assumptions, we come to a very fundamental yet complex question, the arrangement of one sort of element on the other. The word 'order' is so common that we are not conscious of its complexity.

What is the difference between AB and BA? Does it not involve the assumption of a third, perhaps a fourth something? Of two sounds in time can one come first and the other afterwards unless we assume also the idea of a past? Could there be a present and a future without the idea of a past? To get a future must not the present act as past? Is there any difference between the point-pair AB and the point-pair BA apart from their relation to a bearer, a carrier, be this merely time itself? There being no elements but points, and only three of these, can there be a relation among these three points called 'order'?

If the points ABC and no others are on a bearer, they may be said to have order or no order according as the bearer is open or closed. If they have order it may be ABC with CBA, or ACB with BCA, or BAC with CAB, according to the bearer.

There exists a particular geometry, purely qualitative, making no use of the notion of straightness or planeness, but instead only of those of line and surface. This is the so-called *analysis sitûs*. Yet here remains order. Is it not then impossible that without loss of generality order should be subjected to straightness? For simplicity then, for certainty, for accuracy and ease, let us come down from the general idea of order to a more specific idea which we intend to apply only to points on a straight or a straightest, and for which we will take as available the unused word 'betweenness.' Hilbert in 1899 stressed the importance of *between* for the arrangement of costraight points. His treatment was extraordinarily simplified by the elegant proof* of my pupil R. L. Moore that one of his assumptions was redundant. But the problem for points on a straightest is far more difficult. Three terms cannot have a cyclic order, and to say of three points on a circle, that each is between the other two is to waste 'between'. So is the inexpert remark that a point, though it does not divide a straightest into two sects, yet makes of it a single piece in which the points are arranged in a natural order; that is, the words "follow," "precede," "lie between" are applicable.

The great working value of betweenness, is that when a point is known to be between two points, it is thereby located on one particular given straightest. But if we accept 'between' in the above inexpert sense, then to say B is between A and C may mean absolutely nothing, since if A and C are opposite, every other point of the sphere is on a straightest with them and, in the inexpert sense, between them. As a consequence, therefore, the very first of our Assumptions

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*MONTHLY, Vol. IX, April, 1902, pp. 100-1. Cf. E. H. Moore, *Transactions*, Vol. 3 (1902), pp. 142-158.

of betweenness on the sphere, to specify how "between" is to be used of points in a straightest on a sphere, must be:

II 1. No point is between two opposites. This is reinforced by

II 2. Between any two points not opposites there is always a third point.

Filling out this scheme, we have a 'between' that can be used, for example in the definition: Two points A and B, not opposites, upon a straightest a, we call a *sect*, and designate it with AB or BA. The points between A and B are said to be points of the sect AB or also situated *within* the sect AB. The remaining points of the straightest a are said to be situated without the sect AB. The points A, B are called *end-points* of the sect AB.

The next set of assumptions are of congruence. This is a matter which the inexpert suppose they can finish briefly thus:

Definition. Geometrical figures which may be carried over into one another by rigid motions are said to be congruent ($\equiv$).

Theorem. A sect is congruent to itself reversed in direction.

Proof. The point A may be be applied to the point B and the direction AB to the direction BA. Then B will fall upon A: for otherwise the part and the whole would be congruent.

But this treatment is entirely inadmissible.

To define congruence by rigid motion is false and fallacious, since the intuition of rigid motion involves, contains, and uses the congruence idea. We must base the idea of motion on the congruence idea.

A man of whom it has been said: "He was by far the most eminent Amerisan of the Colonial Period, whether we regard the influence of his labors and opinions, upon his own time, in his own country, their wide diffusion in others, or that survival of prestige and authority which yet perpetuates his name and memory," Jonathan Edwards, who died president of Princeton, says, "Motion is a body's existing successively in all the immediate contiguous parts of any distance, without continuing for any time in any one of them."

Its geometric substratum, then, is the preexistence of a series of congruent figures. So rigid motion presupposes congruence.

Moreover, we do not need the troublesome idea 'direction.' In the plane, 'same direction' assumes the whole theory of parallels. On the sphere no two straightests have the same direction, since no two are parallel, yet every two have the same direction, since they go from the same point to the same point. Nor is anything gained by agreeing to call a ray a direction. So preceding motion must come congruence, the idea to be made precise by assumptions.

But right here an unexpected and hitherto unsuspected simplification is possible. In his first congruence axiom, III 1, Hilbert explicitly assumes "Every sect is congruent to itself, that is, always $AB \equiv AB$." Now this assumption is redundant, as was Hilbert's II 4. It is a demonstrable theorem. Two proofs of it have been given me by R. L. Moore, based on the Assumptions of congruence:

III 1. If A not$\equiv B$, and A' not$\equiv C'$, then on ray $A'C'$ there is one and only one point, B', such that $AB \equiv A'B'$.

III 2. If $AB \equiv A'B'$ and $A'B' \equiv A''B''$, then $AB \equiv A''B''$.

III 3. If B is between A and C and B' is between A' and C', and $AB \equiv A'B'$ $BC \equiv B'C'$, then $AC \equiv A'C'$.

We next define angle as two rays from a common point, and assume as

III 4: On a given side of a given ray there is one and only one angle congruent to a given angle.

Instead now of two congruence assumptions

III a: Angles congruent to the same are congruent to each other, and

III b: Euclid I. 4,

we may prove these two as theorems by defining congruent angles in terms of congruent sects, and assuming

III 5: Euclid I. 8: If A, B, C are non-costraight and so are A', B', C', and C is between B and D, and C' is between B' and D', and $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$ and $BD \equiv B'D'$, then $AD \equiv A'D'$.

In itself a point-pair not only has no order, it does not even possess sense. But a sect, a point-pair on a straightest, has sense, and $AB \equiv BA$ must be proved or assumed. It may be assumed without introducing any divergence between the old concept, superposable, and the more fundamental concept, congruent. AB is superposed on BA by a semi-rotation about their common mid-point. On the sphere an angle, the figure two rays from the same initial point, has sense.

The analogue of the semi-rotation of a sect about its mid-point is the semi-rotation of an angle about its mid-ray.

If two figures have central symmetry in a plane, either can be made to coincide with the other by turning it in the plane through two right angles. This holds good when for "plane" we substitute "sphere." Any sect and its inverse, AB and BA, are such figures. They are symcentral about the mid-point of either.

If two figures have axial symmetry in a plane, they can be made to coincide by folding the plane over along the axis, but not by any sliding in their plane. Two axially symmetrical figures in a plane can be brought into coincidence by a semi-revolution of one about the axis. That is, we must use the third dimension of space, and then their congruence depends on the property of the plane that its two sides are indistinguishably alike, any plane will completely fit its trace after being turned over. This procedure, folding over along a line, can have no place in a strictly two-dimensional geometry.

So figures with axial symmetry on a sphere cannot be made to coincide. Let symmetric henceforth mean axially symmetric, and be denoted by $\cdot|\cdot$. A spherical angle and its inverse, $\widehat{\sphericalangle}(h, k)$ and $\widehat{\sphericalangle}(k, h)$ are not symcentral and cannot be brought into coincidence. Should we take as a definition:

An angle is called symmetric to another to whose inverse it is congruent; $\sphericalangle(h, k) \cdot|\cdot \sphericalangle(v, w)$ when $\sphericalangle(h, k) \equiv \sphericalangle(w, v)$, then from $\sphericalangle(h, k) \equiv \sphericalangle(h, k)$ comes $\sphericalangle(h, k) \cdot|\cdot \sphericalangle(k, h)$, but these two are not superposable, cannot be made to coincide.

To those then who have made ideal superposition the basis and test of con-

gruence, the fact that a spherical angle can in no way be placed upon its inverse introduces a radical difference between their presentation of spherics and the familiar presentations of plane geometry. For them many theorems would be bifurcate, for example, Theorem: Any two right angles are either congruent or symmetric. But since congruence in no way depends on the subsequenr concept, motion, nothing is more simple than to assume $\widehat{\measuredangle}(h, k) \equiv \widehat{\measuredangle}(k, h)$.

The three points which determine a triangle have no order and no betweenness, in the specific meaning, for that would make two of a like character and the third of a different character. But as a triad of sects, a triangle has as an individual 'umlaufssinn', tour-sense.

If the congruence of sects is connected with that of angles by a triangle assumption, then if this is restricted to triangles of the same tour-sense, it will not give us the congruence of the basal angles in an isosceles triangle. To attain this is in that case needful the adjunction of one or two continuity assumptions.

Thus we see that beyond the one dimensional space-symmetry implied in the congruence of sect and angle with their inverses, there is a quite distinct two-dimensional space symmetry. This is usually unrecognized in the ordinary treatment of plane geometry, since a plane triangle can have its tour-sense changed by turning it over in the third dimension.

In two-dimensional spherics this is impossible, so although this two-dimensional space-symmetry is presumed in the triangle-assumption's non-recognition of tour-sense, yet it is customary to note its perception in the distinction between congruent and symmetric triangles.

Had this distinction been retained further back, namely for angles, we might have used it in the definition: Two triangles are called symmetric when their corresponding sides are congruent and their corresponding angles are symmetric. But a definition perhaps more desirable comes from setting up the distinction of right side and left side of an angle.

Besides the assumptions of congruence, we need no metric assumptions, and as definition or axiom are well rid of that snaky phrase: "A straight line is the shortest distance between two points." In this reference see Georg Hamel: Ueber die Geometrieen, in denen die Geraden die Kürzesten sind, Math. Ann. Bd. 57, 1903.

Now as to continuity, there seems more call for such assumptions in spherics than in planimetry, since, for example, in the plane a sect may be readily divided into any desired number of parts, while in spherics recourse must be had to a continuity assumption even to show that a given sect has third parts, that one-third of a given sect exists. Nevertheless continuity remains costly. Even Hilbert has not succeeded in attaining a simple treatment of it. His 'Axiom der Vollständigkeit' and 'Axiom der Nachbarschaft' seem inelegant lumps in his beautiful and fine mosaic. Russell says (*Principles of Mathematics*, p. 440): "Whether the axiom of continuity be true as regards our actual space is a question which I see no means of deciding. "For any such question must be empirical, and it would be quite impossible to distinguish empirically what may be

called a rational space from a continuous space."

In my *Rational Geometry* I treat Non-Euclidean Spherics without any continuity assumption whatsoever.

PROBLEMS FOR SOLUTION.

ALGEBRA.

E. Kesner, Salida, Col., solved 208 and 209.

211. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Prove that $p-qx$ and $q-px$ tend to equality as x dimishes to zero, but yet that their limits are not equal. [Edwards' *Differential Calculus*, p. 7, ex. 10.]

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

The limits of $p-qx$ and $q-px$ as x dimishes to zero, are p and q. Suppose $p>q$, then by subtraction the difference at the limit is $p-q$ and the difference for any value of x is $p-q-qx+px$ which evidently decreases as x diminishes. Likewise for $q>p$. Therefore $p-qx$ and $q-px$ tend to equality as x diminishes to zero.

Also solved by G. B. M. Zerr, and J. Scheffer.

212. Proposed by F. P. MATZ, Ph. D., Sc. D.

Solve $\dfrac{x^2+2x+3}{x^2-2x+3}+\dfrac{x^2-2x+3}{x^2+2x+3}=\dfrac{10}{3}$.

I. Solution by O. S. WESTCOTT, Waller High School, Chicago, Ill., ELMER SCHUYLER, Brooklyn, N. Y., and J. SCHEFFER, Hagerstown, Md.

If we write $\dfrac{x^2+2x+3}{x^2-2x+3}=y$, then will $y+\dfrac{1}{y}=\frac{10}{3}$, and $y=3$ or $\frac{1}{3}$.

a) If $\dfrac{x^2+2x+3}{x^2-2x+3}=3$, $x=3$ or 1.

b) If $\dfrac{x^2+2x+3}{x^2-2x+3}=\frac{1}{3}$, $x=-3$ or -1.

II. Solution by A. H. HOLMES, Brunswick, Maine.

Clearing the given equation of fractions and reducing, we obtain $x^4-10x^2=-9$, whence $x=1$, -1, 3, -3.

Also solved by G. W. Greenwood. E. L. Rich, M. E. Graber, G. B. M. Zerr, F. D. Posey, S. S. Flory, L. E. Newcomb,and E. Kesner.

213. Proposed by F. P. MATZ, Ph. D., Sc. D.

Find the two roots of the equation $x^5-209x+56=0$, whose product is unity.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill., and F. D. POSEY, A. B., San Mateo, Cal.

The equation may be written $(x^2-4x+1)(x^3+4x^2+15x+56)=0$. Hence two of the roots are $2\pm\sqrt{3}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let y and $1/y$ be the roots, then

$$y^5-209y+56=0\text{..........}(1),$$
$$56y^5-209y^4+1=0\text{..........}(2).$$

56 times (1)−(2) gives $209y^4-56\times209y+56^2-1=0$, or $209(y^4-56y+15)=0$.

$\therefore y^4-56y+15=0$..........(3).

(1)−56 times (2) gives, after dividing by y, $15y^4-56y^3+1=0$..........(4).

15 times (3)−(4) gives $y^3-15y+4=0$..........(5).

(3)−15 times (4) gives $4y^3-15y^2+1=0$ ....(6).

4 times (5)−(6) gives $y^2-4y+1=0$. $\therefore y=2\pm\sqrt{3}$.

Hence the roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Also solved by A. H. Holmes, J. Scheffer, Lloyd Holsinger, L. E. Newcomb, and Elmer Schuyler.

214. Proposed by Editor EPSTEEN.

How many terms are there in the expansion of $(x_1+x_2+\text{..........}+x_n)^n$?

I. Solution by F. OWENS, M.A.; Mrs. F. OWENS, M.A.; G. B. M. ZERR, A. M., Ph. D.; G. W. GREENWOOD, M. A.

Let ${}_nu_r$ denote the number of terms in the rth power of an expression consisting of n terms. By considering it in the form

$$[x_1+(x_2+x_3+\text{..........}+x_n)]^r$$

we see that

$${}_nu_r=1+{}_{n-1}u_1+{}_{n-1}u_2+\text{..........}+{}_{n-1}u_r.$$

$\therefore {}_nu_r={}_nu_{r-1}+{}_{n-1}u_r$.

Hence we may write down the number of terms in any power of any expression thus:

1	1	1	1	1	....
1	2	3	4	5	....
1	3	6	10	15	....
1	4	10	20	35	....
.....	.....	.....	.....	.....	....

where each element is obtained by adding the one the left to the one above it.

It can be seen, from Pascal's Triangle, that the number of terms is given by the expression for the $(r+1)$st terms of the nth order of figurate numbers, ${}_nu_r=\dfrac{(n+r-1)!}{r!\,(n-1)!}$. In particular ${}_nu_n=\dfrac{(2n-1)!}{n!(n-1)!}$.

II. Solution by J. SCHEFFER, Hagerstown, Md.

The problem reduces to finding the number of products of n quantities and their power. By expanding, $\dfrac{1}{1-x_1x}$, $\dfrac{1}{1-x_2x}$, $\dfrac{1}{1-x_3x}$, etc., into series, we find,

$$\frac{1}{1-x_ix}=1+x_ix+x_i^2x^2+x_i^3x^3+\text{.........}\quad (i=1,\ 2,\ \text{.........},\ n).$$

$\left(\dfrac{1}{1-x_1x}\right)\left(\dfrac{1}{1-x_2x}\right)\left(\dfrac{1}{1-x_3x}\right)\text{.........}=1+s_1x+s_2x^2+s_3x^3+\text{.........}$ where $s_1=x_1+x_2+\text{.........}x_n$, $s_2=x_1^2+x_2^2+x_3^2\text{.........}+x_1x_2+x_1x_3\text{.........}+x_2x_3\text{.........}$, $s_3=x_1^3+x_2^3+\text{.........}x_n^2$ $x_1^2x_2+x_1^2x_3+\text{.........}+x_1x_2x_3+\text{.........}$

To find the number of products in any of these sets of products, we put $x_1=x_2=x_3=\text{.........}=x_n=1$, and we get

$$\frac{1}{1-x_1x}\cdot\frac{1}{1-x_2x}\cdot\frac{1}{1-x_0x}\text{.........}=\frac{1}{(1-x)^n}=(1-x)^{-n}.$$

$\therefore s_n$ is equal to the coefficients of x^n in the expression of $(1-x)^{-n}$.

$\therefore s_n=\dfrac{n(n+1)(n+2)\text{.........}[n+(n-1)]}{n\,!}$, or, multiplying numerator and denominator by $(n-1)\,!$ we get $\dfrac{(2n-1)\,!}{(n-1)\,!n\,!}$ as the required result.

Also solved by L. E. Newcomb, and Elmer Schuyler.

GEOMETRY.

Two solutions of 238 were received from L. E. Newcomb, Los Gatos, Cal.

239. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Divide the sides of a triangle ABC internally in P, Q, R, so that $BP/PC=CQ/QA=AR/RB$. QR cuts BC externally in S. Show that BS is to CS in the duplicate ratio of CP to PB.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Draw SD parallel to AB; let m=any proper fraction, and let $PC=ma$, $QA=mb$, $RB=mc$. Then

$$\frac{BP}{PC}=\frac{a(1-m)}{am}=\frac{CQ}{QA}=\frac{b(1-m)}{bm}=\frac{AR}{RB}=\frac{c(1-m)}{cm}=\frac{1-m}{m}.$$

Considering AB, AC as the axes of coördinates the equation to RQ is

$$\frac{x}{bm}+\frac{y}{c(1-m)}=1,\text{ equation to } BC \text{ is } \frac{x}{b}+\frac{y}{c}=1.$$

$$\therefore x = AD = \frac{bm^2}{2m-1},\ y = DS = \frac{c(1-m)^2}{2m-1} = cn.$$

$$CD = AD - b = \frac{b(1-m)^2}{2m-1} = bn.$$

$CS = n\sqrt{(b^2+c^2-2bc\cos A)} = an$ since $\angle CDS = A$.
$BS = a + CS = a(n+1)$.

$$\therefore CS = \frac{a(1-m)^2}{2m-1},\ BS = \frac{am^2}{2m-1}.\quad \therefore BS : CS = m^2 : (1-m)^2.$$

$CP^2 = a^2m^2,\ BP^2 = a^2(1-m)^2$.
$\therefore CP^2 : BP^2 = m^2 : (1-m)^2,\ \therefore BS : CS = CP^2 : BP^2$.

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Through C draw a parallel to RQ which will intersect AB in a point M, say, then

$$\frac{BS}{CS} = \frac{RB}{MR}.\ \text{But}\ \frac{MR}{CQ} = \frac{AR}{QA}.\ \therefore \frac{BS}{CS} = \frac{RB}{AR}\cdot\frac{QA}{CQ} = \left(\frac{PC}{PB}\right)^2.$$

III. Solution by J. SCHEFFER, Hagerstown, Md.

Draw CF parallel to AB, meeting RQ in T.

From $\dfrac{PB}{PC} = \dfrac{QC}{QA} = \dfrac{RA}{RB}$, we have $PB.QA = PC.QC$, $PB.RB = PC.RA$.

By multiplication, $PB^2.QA.RB = PC^2.QC.RA$, or $\dfrac{PB^2}{PC^2} = \dfrac{QC.RA}{QA.RB}$ ..... (1).

But $\dfrac{RA}{CT} = \dfrac{QA}{QC}$, and $\dfrac{CT}{RB} = \dfrac{CS}{BS}$; $\therefore \dfrac{RA}{RB} = \dfrac{QA}{QC}\cdot\dfrac{CS}{BS}$.

Substituting in (1), $\dfrac{PB^2}{PC^2} = \dfrac{CS}{BS}$.

240. Proposed by B. F. BURLESON.

The points P_1, P_2, P_3 in the perpendicular BD of an isosceles triangle with equal angles at A and C are at the intersection of the three perpendiculars of the triangles, the center of its inscribed, and the center of its circumscribed circles, respectively. The distance from P_1 to P_2 is $m=16$ rods, and the distance from P_2 to P_3 is $n=17$ rods. Required the radii R and r of the triangle's circumscribed and inscribed circles, the perpendiculars $BD = P_b$, the base $AC = 2b$, and one of the equal sides as $AB = a$.

I. Solution by the PROPOSER.

We derive from Chauvenet's *Trigonometry*, equation 300 and equation 298,

$$R^2-2Rr=n^2\dots\dots\dots(1),$$
$$R^2-4r^2=(m+n)^2-2m^2\dots\dots\dots(2).$$

Resolving (1) and (2), we find

$$R=\frac{n^2}{n-m}=289 \text{ rods, and } r=\frac{m(2n-m)}{2(n-m)}=144 \text{ rods.}$$

We have in any isosceles triangle

$$P_b=\frac{2rb^2}{b^2-r^2}=\frac{4n^2-m^2}{2(n-m)}=450 \text{ rods}\dots\dots(3). \quad \text{Also } P_b=\surd[(R^2-b^2)+R]\dots\dots(4).$$

Equating (3) and (4) and solving, we find that

$$2b=AC=2\surd\{r[2R-r-2\surd(R^2-2Rr)]\}=\frac{n}{n-m}\surd(4n^2-m^2)=480 \text{ rods.}$$

Whence $a=AB=\surd(P_b{}^2+b^2)=\frac{n}{n-m}\surd(4n^2-m^2)=510$ rods.

II. **Solution by A. H. HOLMES, Brunswick, Me.**

We have $BD=\surd(a^2-b^2)$, $r=\frac{b\surd(a^2-b^2)}{a+b}$, and $R=\frac{a^2}{2\surd(a^2-b^2)}$,

$DP_1=\frac{b^2}{\surd(a^2-b^2)}$, $DP_2=r$, and $DP_3=\frac{a^2-2b^2}{2\surd(a^2-b^2)}$.

$$\therefore \frac{b\surd(a^2-b^2)}{a+b}-\frac{b^2}{\surd(a^2-b^2)}=m=16\dots\dots\dots(1),$$

$$\frac{a^2-2b^2}{2\surd(a^2-b^2)}-\frac{b\surd(a^2-b^2)}{a+b}=n=17\dots\dots\dots(2).$$

From (1), $\surd(a^2-b^2)=\frac{b}{m}(a-2b)$.

From (2), $\surd(a^2-b^2)=\frac{a}{2n}(a-2b)$. $\therefore a=\frac{2n}{m}b$.

Putting the value of a in (1), $b=\frac{m}{2(n-m)}\surd(4n^2-m^2)$.

$$\therefore a=\frac{n}{n-m}\surd(4n^2-m^2),\ BD=\frac{4n^2-m^2}{2(n-m)},\ r=\frac{m(2n-m)}{2(n-m)},\ R=\frac{n^2}{n-m}.$$

When $m=16$ and $n=17$, the equal sides become 510, base=480, perpendicular=450, $r=144$, $R=289$.

Also solved by G. B. M. Zerr, J. Scheffer, L. E. Newcomb.

241. Proposed by Editor EPSTEEN.

If two conics have each double contact with a third, their chords of contact with that conic, and two of the lines through their common points, will meet in a point and form a harmonic pencil.

I. Solution by F. D. POSEY, A. B., San Mateo, Cal.

Let $S=0$ be the equation of the conic which has double contact with the first conic in the line $\alpha_1=0$ and with the second conic in $\alpha_2=0$. The equation of these two conics may then be written

$$c-\lambda_1^2\alpha_1^2=0 \text{.........}(1),\quad c-\lambda_2^2\alpha_2^2=0\text{..........}(2).$$

The two straight lines $\lambda_1^2\alpha_1^2-\lambda_2^2\alpha_2^2=0$ go through the common points of (1) and (2), through the intersections of $\alpha_1=0$ with $\alpha_2=0$, and it is evident that the lines $\alpha_1=0$, $\lambda_1\alpha_1-\lambda_2\alpha_2=0$, $\alpha_2=0$, $\lambda_1\alpha_1+\lambda_2\alpha_2=0$, form a harmonic pencil.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $S=0$ be the equation to the third conic. Then $S=k\alpha^2$, $S=k_1\beta^2$ are the equations to the other two conics (Salmon's *Conic Sections*, Art. 252, page 234). Subtracting, $\alpha^2-\beta^2=0$ or $(\alpha-\beta)(\alpha+\beta)=0$. This represents a pair of chords of intersection passing through the intersection of the chords of contact $\alpha=0$, $\beta=0$. Also $\alpha=0$, $\beta=0$, $\alpha-\beta=0$, $\alpha+\beta=0$ form a harmonic pencil. (See Salmon's *Conic Sections*, Art. 263, page 242).

Also solved by G. W. Greenwood.

242. Proposed by the late MARCUS BAKER.

In a trapezoid $ABCD$, upper base $BC=a$, lower base $AD=b$, a line CP is drawn from vertex C to any point P in the base AD, such that $PD=mb$. The line CP intersects the diagonal BP in M and MN is drawn parallel to the bases meeting CD in N; then is $M.N=\dfrac{abm}{a+bm}$.

Solution by the PROPOSER, and G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

From the similar triangles DMN, DBC, we have

$$\frac{MN}{a}=\frac{MD}{BD}.$$

From the similar triangles MPD, MBC, we have

$$\frac{MD}{BD}=\frac{mb}{a+mb}.\qquad \therefore MN=\frac{amb}{a+mb}.$$

Also solved by A. H. Holmes, G. B. M. Zerr, F. D. Posey, J. Scheffer, and L. E. Newcomb.

CALCULUS.

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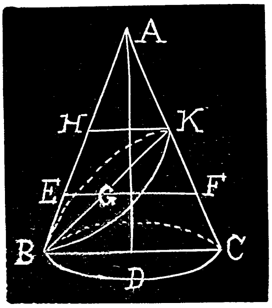
185. Proposed by A. H. HOLMES, Brunswick, Maine.

Required the perpendlcular height of a right cone, radius of base being unity, such that the maximum ellipse that can be cut from the cone shall equal the base of the cone in area.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $BC=2=c$; a, b the semi-axes of the ellipse, $\angle KBC=\theta$, $\angle DAC=\beta$. Then $BK=2a=c\cos\beta/\cos(\theta-\beta)$ or $a=\frac{1}{2}c\cos\beta/\cos(\theta-\beta)$, $b^2=EG.GF=\frac{1}{4}cHK$. But $HK=2a\cos(\theta+\beta)/\cos(\theta-\beta)$.



$\therefore b=\frac{1}{2}c\surd[\cos(\theta+\beta)/\cos(\theta-\beta)]$, and therefore $\pi ab=\frac{1}{4}\pi c^2\cos\beta\surd[\cos(\theta+\beta)]/[\cos(\theta-\beta)]^{\frac{3}{2}}=$maximum.

$\therefore \dfrac{\surd[\cos(\theta+\beta)]}{\cos(\theta-\beta)\surd[\cos(\theta-\beta)]}=$maximum, and $\sin2\theta=2\sin2\beta$, or, $\sin\theta=\frac{1}{2}\surd\{2[1+\surd(1-4\sin^2 2\beta)]\}$.

Since $c=2$ and $\pi ab=\pi c^2/4$, we have $\cos^2\beta\cos(\theta+\beta)=\cos^3(\theta-\beta)$. Now $\cos\theta=\frac{1}{2}\surd\{2[1-\surd(1-4\sin^2 2\beta)]\}$,

$$\cos^3\beta\sin\theta\cos\theta=4\sin\beta\cos^2\beta\cos^2\theta+3\sin^2\beta\cos\beta\sin\theta\cos\theta+\sin\beta\sin^2\theta.$$

Substituting the values of $\sin\theta$, $\cos\theta$ and reducing,

$$4\sin^2 2b+1=(2\cos2\beta+1)\surd(1-4\sin^2 2\beta).$$

$\therefore 16\sin^4 2\beta+40\sin^2 2\beta=7$, and $\sin2\beta=\frac{1}{2}\surd(4\surd2-5)$; $\beta=11^\circ\ 57'\ 9''$.

$\therefore$ Altitude$=\cot\beta=4.72388$.

Also solved by J. Scheffer, the Proposer, and L. E. Newcomb.

186. Proposed by Editor EPSTEEN.

Evaluate $\displaystyle\int_0^\infty\frac{\sin my}{y}dy$, $\displaystyle\int_0^\infty\frac{\cos my}{y}dy$.

I. Solution by IDA M. SCHOTTENFELS, M. A., New York City.

$\displaystyle\int_0^\infty\frac{\sin my}{y}dy=+\frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$, according as $m>0$, $m=0$, $m<0$ if $y>0$.

Proof. If $y>0$ then $\displaystyle\int_0^\infty e^{-yx}dx=1/y$. Multiplying both members by $\sin my$:

$$(1)\quad \sin my\int_0^\infty e^{-yx}dx=\frac{\sin my}{y};\qquad (2)\quad \int_0^\infty e^{-yx}\sin my\,dx=\frac{\sin my}{y}.$$

Integrating (2) with respect to y between 0 and ∞,

$$(3) \qquad \int_0^\infty \left(\int_0^\infty e^{-yx}\sin my\, dx\right)dy = \int_0^\infty \frac{\sin my}{y}dy.$$

Changing the order of integration since the limits are the same we have

$$\int_0^\infty \left(\int_0^\infty e^{-yx}\sin my\, dy\right)dx = \int_0^\infty \frac{\sin my}{y}dy.$$

But

$$(5) \qquad \int_0^\infty e^{-yx}\sin my\, dy = \frac{m}{m^2+x^2} \quad (y>0).$$

The result in (5) is obtained by finding the indefinite integral and passing to the limit thus,

$$\int e^{-yx}\sin my\, dy = -e^{-yx}\left[\frac{m\cos my + x\sin my}{x^2+m^2}\right]$$

$$\lim_{y \doteq \infty} -e^{-yx}\left[\frac{m\cos my + x\sin my}{x^2+m^2}\right] = \frac{m}{x^2+m^2} \quad (y>0)$$

$$(6) \qquad \int_0^\infty \frac{m}{x^2+m^2}dy = \int_0^\infty \frac{\sin my}{y}dy.$$

$$(7) \qquad \int_0^\infty \frac{m}{x^2+m^2}dx = \tan^{-1}\frac{x}{m}\Big]_0^\infty.$$

$\tan^{-1}\frac{x}{m}\Big]_0^\infty = \frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$ according as $m>0$, $m=0$, $m<0$, and therefore $\int_0^\infty \frac{\sin mx}{x}dx$ has the same values and equation (1) is true.

We now regard the given integral as a function of the parameter m thus

$$\int_0^\infty \frac{\sin my}{y}dy = \varphi(m)$$

and trace the resulting curve.

For all negative values of m the integral $\int_0^\infty \frac{\sin mx}{x}dx$ describes a line parallel to the m axis and at a distance $-\frac{1}{2}\pi$ from it until m reaches the value 0; here the function has a finite discontinuity and jumps from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, and for positive values of m continues in a straight line parallel to the m axis and at a distance $+\frac{1}{2}\pi$ from it.

In a similar manner $\int_0^\infty \frac{\cos my}{y}dy = \infty$ for real finite values of m.

II. Solution by S. A. COREY, Hiteman, Iowa.

We have $1/y=\int_0^\infty e^{-ay}da$ if $y>0$. Hence

$$\int_0^\infty \frac{\sin my}{y}dy=\int_0^\infty \sin my\int_0^\infty e^{-ay}da\ dy=\int_0^\infty\int_0^\infty \sin my\ e^{-ay}\ da\ dy,$$

$$=\int_0^\infty\int_0^\infty \sin my\ e^{-ay}\ dy\ da=\int_0^\infty \frac{mda}{a^2+m^2}$$

$$=\tfrac{1}{2}\pi \text{ if } m>0,\ =-\tfrac{1}{2}\pi \text{ if } m<0,\ =0 \text{ if } m=0.$$

(See Byerly's *Integral Calculus*, page 100.) Similarly,

$$\int_0^\infty \frac{\cos my}{y}dy=\int_0^\infty \cos my\int_0^\infty e^{-ay}\ da\ dy=\int_0^\infty \frac{ada}{a^2+m^2}=0,$$

for all finite real values of m.

III. Remark by F. D. POSEY, A. B., San Mateo, Cal.

A discussion of the first of these integrals will be found on page 271, Art. 285 of Todhunter's *Integral Calculus*. The result there obtained is $\frac{1}{2}\pi$ if m be positive, $-\frac{1}{2}\pi$ if m be negative.

Also solved by G. W. Greenwood, G. B. M. Zerr, and J. Scheffer.

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Ph. D., Sc. D.

Given $\left\{\begin{matrix} a\cos\alpha+b\sin\alpha=c \\ a\cos\beta+b\sin\beta=c\end{matrix}\right\}$ to prove that

$$\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2},\ \text{and}\ \cot\alpha+\cot\beta=\frac{2ab}{c^2-a^2}.$$

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

We have at once $\dfrac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}=-b/a$ .... (1).

Now $\cos\alpha-\cos\beta=-2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$ and $\sin\alpha-\sin\beta=2\cos\frac{1}{2}(\alpha+\beta)\times\sin\frac{1}{2}(\alpha-\beta)$. Substituting these values in (1) we obtain $\tan\frac{1}{2}(\alpha+\beta)=b/a$. Since $\tan\frac{1}{2}(\alpha+\beta)=\pm\sqrt{\dfrac{1-\cos(\alpha+\beta)}{1+\cos(\alpha+\beta)}}=b/a$, and solving,

$$\cos(\alpha+\beta)=\frac{a^2-b^2}{a^2+b^2}\text{.........(2)};\ \sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}\text{.........(3)}.$$

Multiplying together the given equation and substituting the values of $\cos(\alpha+\beta)$,

$\sin(\alpha+\beta)$ from (2) and (3) we find that $\sin\alpha\sin\beta=\dfrac{c^2-a^2}{a^2+b^2}$; and therefore

$$\cot\alpha+\cot\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}=\frac{2ab}{c^2-a^2}.$$

II. Solution by L. S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

Subtracting the second equation from the first, $a(\cos\alpha-\cos\beta)+b(\sin\alpha-\sin\beta)=0$. Hence, $\sin\frac{1}{2}(\alpha-\beta)[a\sin\frac{1}{2}(\alpha+\beta)-b\cos\frac{1}{2}(\alpha+\beta)]=0$. Either $\sin\frac{1}{2}(\alpha-\beta)=0$ or $a\sin\frac{1}{2}(\alpha+\beta)=b\cos\frac{1}{2}(\alpha+\beta)$. In the first case we have the trivial solution $\alpha=\beta$. In the second case, we put $(\alpha+\beta)=x$, $\sin\frac{1}{2}x=b/a\cos\frac{1}{2}x$. Since $\sin x=2\sin\frac{1}{2}x\cos\frac{1}{2}x$ the latter equation takes the form $(a^2+b^2)\sin^2x=2ab\sin x$, whence, excluding the value $\sin x=0$ introduced by squaring, $\sin(\alpha+\beta)=\dfrac{2ab}{a^2+b^2}$.

Secondly, $\cot\alpha+\cot\beta=\dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$. From the first of the given equations $a\sqrt{(1-\sin^2\alpha)}+b\sin\alpha=c$. Solving for $\sin\alpha$, we have $\sin\alpha=\dfrac{bc\pm a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}$

Similarly, solving the second of the given equations, $\sin\beta=\dfrac{bc\pm a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}$

Since the solution $\alpha=\beta$ was rejected, $\sin\alpha\sin\beta=\dfrac{c^2-a^2}{a^2+b^2}$.

Hence $\cot\alpha+\cot\beta=\dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$ becomes equal to $\dfrac{2ab}{c^2-a^2}$.

III. Solution by A. H. HOLMES, Brunswick, Maine.

From $a\cos\alpha+b\sin\alpha=c$,

$$\sin\alpha=\frac{bc+a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2},\quad \cos\alpha=\frac{ac-b\sqrt{(a^2+b^2-c^2)}}{a^2+b^2},$$

and from $a\cos\beta+b\sin\beta=c$,

$$\sin\beta=\frac{bc-a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2},\quad \cos\beta=\frac{ac+b\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}.$$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$. Putting for $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$ their values above, and reducing, $\sin(\alpha+\beta)=\dfrac{2ab}{a^2+b^2}$, $\cot\alpha+\cot\beta=\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\cos\beta}{\sin\beta}$.

Introducing the values of these sines and cosines, and reducing, $\cot\alpha+\cot\beta=\dfrac{2ab}{c^2-a^2}$.

Also solved by G. B. M. Zerr, G. W. Greenwood, J. Scheffer, and E. L. Rich.

PROBLEMS FOR SOLUTION.

ALGEBRA.

219. Proposed by Dr. SAUL EPSTEEN, The University of Chicago.

Sum to infinity $\frac{1.2}{3}+\frac{2.3}{3^2}+\frac{3.4}{3^3}+\ldots\ldots\ldots$

220. Proposed by L. ROBINSON, B. S., Philadelphia, Pa.

Find the sum of the first $n+1$ terms of the series

$$1+\frac{m}{1!}+\frac{m\,(m+1)}{2!}+\frac{m(m+1)(m+2)}{3!}+\ldots\ldots\ldots$$

221. Proposed by F. P. MATZ, Ph. D., Sc. D.

Eliminate the unknowns from

$$x/y+y/z+z/x=a \ldots\ldots\ldots(1),\quad x/z+y/x+z/y=b \ldots\ldots\ldots(2),$$
$$(x/y+y/z)(y/z+z/x)(z/x+x/y)=c \ldots\ldots\ldots(3).$$

222. Proposed by G. W. WALKER, Camden, N. J.

Extract the square root of $87-12\sqrt{42}$.

GEOMETRY.

245. Proposed by J. H. M. MACLAGAN-WEDDERBURN, M. A., Chicago, Ill.

Given two lines of fixed length, $AB=a$, $BC=b$, perpendicular to each other. A line CP is drawn making $\angle BPC=\theta$. AD is the perpendicular to AB meeting CP in D. Find by Euclidean construction the angle θ such that $AD^2\cos^2\theta+b^2\sin^2\theta$ is a minimum.

246. Proposed by T. L. CROYES, Paris, France.

Given a movable point O on a fixed diameter of a circle S, an inscribed triangle ABC, and the perpendiculars OM, ON, OP from the point O on the sides AB, AC, BC. Prove, by pure geometry, that the circle circumscribing the triangle MNP will always pass through a fixed point.

247. Proposed bv SETH PRATT, C. E., Tecumseh, Neb.

From two given points without a circle to draw two lines meeting in the circumference and making equal angles with the tangent at that point.

AVERAGE AND PROBABILITY.

131. Proposed by F. P. MATZ. Ph. D., Sc. D.

In a given square an arc is described at random the center being one of the vertices of the square. What is the probability that this arc is longer than a side of the square?

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

Mr. J. R. Lucky has been appointed assistant in mathematics at Oberlin College.

Mr. R. G. D. Richardson has been appointed instructor in mathematics in Yale University.

Dr. K. Schmidt has been appointed professor of mathematics in the University of Florida.

Prof. Alois F. Kovarik has been appointed instructor in physics in the University of Minnesota.

The first part of the French edition of the *Encyklopädie der Mathematischen Wissenschaften* has appeared.

Dr. Harvey C. DeMotte, professor of mathematics in the Illinois Wesleyan University, died on December 16, 1904.

Dr. H. B. Evans has been promoted to an assistant professorship in mathematics in the University of Pennsylvania.

Prof. F. S. Luther, professor of mathematics in Trinity College, Hartford, Conn., has been elected president of the institution.

The editors are indebted to Mr. H. B. Leonard for assistance in compiling the index of Vol. XI of the MONTHLY, which will appear with the February number.

Dr. J. Stebbins has been promoted to an assistant professorship in mathematics and Mr. A. H. Wilson has been appointed to an instructorship in the University of Illinois.

Dr. R. S. Woodward, professor of mechanics and mathematical physics, and Dean of the School of Pure Science, Columbia University, has been elected president of the Carnegie Institution of Washington, Dr. Gillman having resigned on his 70th birthday.

Dr. F. H. Safford, of the University of Pennsylvania, has recently completed an exhaustive discussion of the seating of seven persons at a round table (Problem 99, March, 1899, March, 1900, and April, 1904). All possible arrangements have been tabulated, with no assumptions concerning groups, and all solutions found have been reduced to a single form, the one previously given.

A mathematical section of the California Teachers' Association was organized on December 26, 1904, at San Jose. G. A. Miller, Stanford University, was elected as president, and J. F. Smith, Campbell High School, as Secretary. The main object of the Association is to arouse more interest in mathematical pedagogy by means of separate meetings for the discussion of recent mathematical movements.

David Eugene Smith, professor of mathematics in Teachers' College, Columbia University, addressed the students of the State Normal School at Plattsburg, N. Y., on January 19, 1905. The subject of his lecture was the History of Mathematics. Professor Smith's recent monograph, "The Outlook for Arithmetic in America," is a valuable contribution to pedagogical literature. As a practical exposition of the author's theories, his two new arithmetics (Ginn & Co., publishers) are attracting wide attention.

BOOKS.

An Introduction to Projective Geometry and Its Applications. An Analytic and Synthetic Treatment. By Arnold Emch, Ph. D., Professor of Graphics and Mathematics in the University of Colorado. 8vo. Cloth, vii+267 pages, 114 figures. Price, $2.50. New York: John Wiley & Sons.

In contrast to the usual presentation of projective geometry in a purely systematic form with little attention given to applications, the present book treats the subject with a view to utility, considerable space being given to practical applications.

In addition to the usual subjects treated in elementary treatises, two chapters on pencils and ranges of conics, including cubics, and on applications of mechanics have been added. As an example of the power of projective geometry, the Stinerian Transformation of the Cubic, treated in Chapter IV, may be cited, and as a particular novel feature of the work, the realization of collineations by linkages, described in Chapter V, may be mentioned. A departure from fairly well established conventional notations occasionally occurs. For example, page 172, the conics whose equations are $u=0$ and $u_1=0$ are referred to as the conics U and U_1. It seems to me to be better to refer to the geometric entities by their algebraic representatives, viz. u and u_1, thus avoiding the danger of taking U and U_1 to be different from u and u_1 when they are intended to be the same.

However, the whole subject as treated by the author is unusually clear and well adapted to the needs of the student of mathematics as well as to the practical mathematician. B. F. F.

The Foundations of the Euclidean Geometry as Viewed from the Standpoint of Kinematics. A Dissertation submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy. By I. E. Rabinovitch. 8vo. Paper Cover, 116 pages. Published by the author.

The first thirty-seven pages of this thesis is devoted to a resumé and comparison of the researches of the various noted mathematicians who have considered the subject of Non-Euclidean Geometry. The remaining part of the thesis is devoted to the investigations of the Foundations of Geometry from the kinematical standpoint. This thesis is certainly a very valuable contribution to the science of mathematics. B. F. F.

The Elements of Analytical Geometry. By Percy F. Smith, Ph. D., Professor of Mathematics in the Scheffield Scientific School of Yale University, and Arthur S. Gale, Ph. D., Instructor in Mathematics in Yale College. 8vo. Cloth, xii+424 pages. Price, $2.00. Boston and Chicago: Ginn & Co.

A glance at the table of contents of this volume will show that a number of topics not usually found in elementary treatises are here discussed, some of which are the invariant properties of the equation of the second degree and Inversion. The whole subject is admirably treated. B. F. F.

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SOME NEW RATIOS OF CONIC CURVES.

By ALAN S. HAWKESWORTH.

[Concluded from January Number.]

Theorem 9. The focal chords through corresponding points [p and P] upon reciprocal curves determine at their other extremities two other corresponding points [q and Q]; and are thus corresponding chords [Figure 2].

Let p on the ellipse, and P on the hyperbola be the corresponding points, determined by the common radius $S'pEP$. And let pfq and PSQ be the respective focal chords. Then will qQ be also corresponding points, and have the determining radius $S'qFQ$ in common.

For, angles pfS' and PSd are equal [Theorem 6]; and hence also angles qfS' and QSd; or qfa and QSA; which angles must therefore subtend corresponding points [Theorem 6, Corollary 1], upon a common determining radius $S'qFQ$. And similarly, if $p'fq'$ and $P'Q'S$ be the corresponding focal chords; they are determined by the common radius $S'q'Q'$, and diameter $p'S'P'$ [Figure 3]. While, if focal chords through S' are chosen, still more obviously is this true; for the said focal chords and their determining radius or diameter now coincide.

Corollary 1. If lines Ef and Ff, ES and FS be bisected in rr' and RR' respectively, by the several auxiliary circles [Theorem 2], then these points $rr'R'R$ form a rhomboid. For rR and $r'R'$ are fixed, in both magnitude and direction, being parallel to, and one half of fS [Theorem 6, Corollary 3]; while EF, the director chord, which is similarly parallel to, and double of rr', and RR', is clearly variable, both in length and direction.

Theorem 10. The three tangents of the three corresponding points upon

the two reciprocal curves and their common director circle meet each other upon the common directrix. So that the six tangents of two corresponding focal chords, and of the director chord of their two determining radii, will meet in one such point upon the common directrix [Figure 2].

Let *pq* and *PQ* be the two corresponding focal chords; and *EF* their common director circle, between *E* and *F*, the extremities of their two determining radii *S'pEP* and *S'qFQ*.

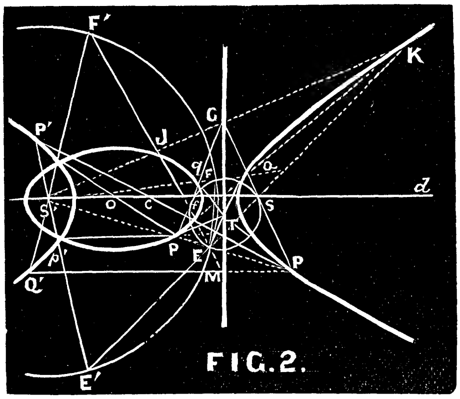


FIG. 2.

Draw, say *PT* and *QT*, the tangents of the focal chord *PSQ* of the hyperbola; meeting in *T* upon the directrix. Join *TE*, *Tp*, *Tq*, *TF*, *Tf*, and *TS*. Then will *Tq* and *Tp* be the tangents to the focal chord *pfq* of the ellipse; and *TE*, *TF* the two tangents at *E* and *F* to the director circle.

For angles *TPE* and *TPS* are equal; sides *PE* and *PS* are equal; and *PT* is common; so that triangle *PET* is equal in all respects to the right angled triangle *PST*. And thus *TE* is both the tangent at *E* to the director circle, and is equal to *TS*. Similarly, *TF* is its tangent at *F*, and is equal to *TS*. Now *fX* equals *XS* [Theorem 7]; and *XT* being a perpendicular, *fT* equals *ST*; and therefore also *TE* and *TF*. Hence the two triangles *pET* and *pfT* are equal in all respects; for they have three sides of each equal; *pE* to *pf*; *ET* to *Tf*; and *pT* common. So that *pET* and *pfT* are right angles; and *pT* bisects externally the angle *S'pf*; and is thus the tangent at *p* to the ellipse. Similarly, *qT* is its tangent at *q*. And since the tangents of a focal chord, in any conic curve, meet each other upon the directrix; therefore *T* is the meeting point of the six tangents *PT*, *QT*, *ET*, and *FT*, *qT* and *pT*.

Were *P'Q'S* and *p'fq'* the focal chords chosen [Figure 3], with their director chord *JK*; then in like manner it can be shown that their respective tangents meet in one and the same point upon the common directrix.

Corollary 1. But, if it be focal chords through *S'*—as *e. g.* *S'pP*—with their resultant director chord identical with the common determining diameter [*S'E*]; then, while the three tangents at *p*, *P*, and *E* meet, as we have seen, in point *T* on the directrix; yet this same point cannot now be also the meeting point of the other three tangents of ellipse, hyperbola, and circle. For, first, the two tangents of a focal chord through *S'* evidently meet, not upon *XT*, the common directrix of foci *f* and *S*, but upon the respective directrix of *S'*; while, secondly, the tangent to the circle, on the opposite extremity of the director chord diameter through *E*, must be parallel to that at *E*. And thus the second set of three tangents must meet in a second point *T'* upon the common directrix *XTT'*.

Corollary 2. Point *T*, the meeting point of the tangents of the focal chords through *f* and *S*, is clearly also the intersection of the directrix *TX* by

the radius which perpendicularly bisects the director chord; even as it is also its intersection by the semi-diameter of ellipse or hyperbola, respectively bisecting the focal chords.

Corollary 3. If, then, with this point T as center, and radius Tf or TS, a circle be described [Figure 2], it will ever circumscribe the simple quadrangle $fESF$, formed by joining the respective foci f and S to the ends of the director chord.

While, on the other hand, it will ever be circumscribed by the quadrilateral $pPQq$; formed by joining the extremities of the two focal chords; and tangential to it at points $fESF$, where the major axis, and director chord, respectively, cut the sides pq, pP, PQ, and qQ; which points evidently coincide with the four corners of the inscribed quadrangle.

And furthermore, this circle will be either inscribed, or escribed to the said quadrilateral $pPQq$, according as PQ, the focal chord of the hyperbola, cuts branch S alone, or both the branches S and S'; thus causing the quadrilateral to be either simple—*e. g.* $pPQq$ [Figure 2]—; or re-entrant—as *e. g.* $p'P'Q'q'$ [Figure 3].

If, however, the chosen focal chords be through S'; with resultant double points T and T', then no circle can be drawn, since all the eight points, corresponding to $pqPQEFf$ and S, now lie collinearly upon the common determining diameter; which is also both focal and director chords.

Theorem 11. If h, upon the major axi, be the point of intersection by the director chord EF, and thus a summit of the quadrangle $fESF$ inscribed to the circle, whose radius is Tf or TS; it will also be the point of intersection of two diagonals pQ and qP of the quadrilateral $pPQq$, circumscribing that same circle, and tangential to it at points $fESF$ [Figure 2].

For the four points $fESF$, being thus common to all three figures—*i. e.* the circle about T, its inscribed quadrangle, and its circumscribing quadrilateral —they must therefore possess in common the self-conjugate triangle hJK.

Corollary 1. Thus J and K are also points in common, being both the other two summits of the inscribed quadrangle, and also the other two intersections of diagonals PqJ, pQK, and $S'JK$, to the circumscribing quadrilateral $pPQq$. So that lines Ef, SF, and Pq are ever concurrent; as also ES, fF, and pQ; and concurrent, too, on a common radius $S'JK$.

Theorem 12. Therefore, even as the fixed points f and S of the two reciprocal curves harmonically divide the diameter [$2S'D$] of their common director circle; so also do any two corresponding points upon the curves harmonically divide the common radius or diameter determining them.

For, taking the circumscribing quadrilateral $pPQq$ [Figure 2], and considering it also as a quadrangle, then point h, where the director chord cuts the major axi [Theorem 11] will not only be the internal intersection of the diagonals pQ and qP; but also one of its three summits, of which the other two are S' and G. While similarly, taking the inscribed quadrangle $fESF$, and considering it also as a quadrilateral, since its three summits coincide with the three in-

tersections of the diagonals of $pPQq$ [Theorem 11, Corollary 1]; so conversely, must the intersections of its three diagonals—*i. e.* S', G, and h—coincide with the three summits of $pPQq$.

And thus the pencil ray of Gh must pass through F and E. So that $S'p : S'P = Ep : EP$; and $S'q : S'Q = Fq : FQ$, etc. While, since corresponding focal chords can be drawn through any two corresponding points—such as $p'q'$ and $P'Q'$; or $p'S'P'$—we can in like manner prove that $S'p' : S'P' = Jp' : JP'$ [Figure 3].

Theorem 13. If pm and PM be the perpendiculars from the reciprocal points p and P respectively, to their common directrix; and E be the extremity of their common determining radius $S'pEP$; then will lines EM and Em pass through f and S, the fixed points [Figure 2].

For $pE : pm = fp : pm = fa : aX = S'f : a'a = S'f : S'D = S'D : S'S = S'E : S'S$; while pm is parallel to $S'S$; and thus angles Epm and $ES'S$ are equal. Therefore Epm and $ES'S$ are similar triangles; and EmS one right line. And in a like manner, $PM : PE = PM : SP = AX : SA = A'A : S'S = S'D : S'S = S'f : S'D = S'f : S'E$; while angles MPE and $ES'f$ are equal. Therefore MEP and fES' are also similar triangles; angles MEP and fES' equal; and thus MEf one right line.

If Q' be taken on the other branch of the hyperbola, with $Q'S'q'F'$ for its determining diameter; and $Q'M'$, and say $q'm'$ for its perpendiculars to the directrix; then similarly $F'fM'$ will be one right line; and also $F'm'S$.

Corollary 1. pEm and PME, like $S'ES$ and $S'fE$, are similar triangles; and likewise $q'F'm'$ and $Q'M'F'$; $S'F'S$ and $S'fF'$.

Theorem 14. The corresponding chords of reciprocal curves, and their common director chord, all three meet in one point upon their common directrix [Figures 2 and 3].

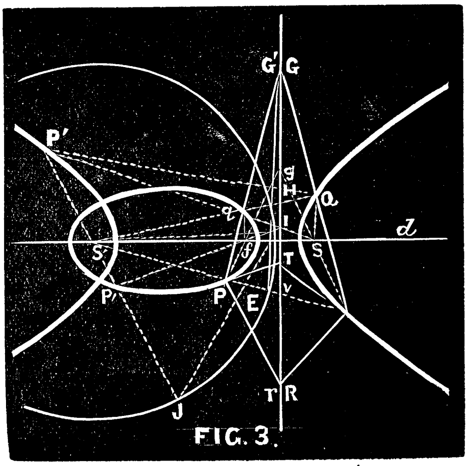


FIG. 3.

First, let the reciprocal focal chords pq and PQ, and their director chord EF, determined by the two radii $S'pEP$ and $S'qFQ$, be the three chords in question [Figure 2], and let pq cut the directrix in, say g; and PQ cut it in G; we will first prove g and G to be identical.

For angles qfa, or gfX, and QSA, or GSX, are equal [Theorem 6]; while fX equals SX [Theorem 7]. So that the right angled triangles fXg and SXG are equal in all respects; Xg to XG, and thus g and G are the same point. Therefore the reciprocal focal chords pqG and PQG meet in the same point G upon the directrix.

Next, G being thus a summit; S' another summit; and $S'pEP$, $S'qFQ$, harmonic ranges upon $pPQq$ considered as a quadrangle [Theorem 12], it follows that GS', Gqp, GFE, and GQP form a harmonic pencil. So that the director chord EF meets its reciprocal focal chords pq and PQ also in G upon the directrix.

And similarly, if $p'fq'I$ and $P'IQ'S$ be the two focal chords chosen [Figure 3], with their common director chord JK, determined by radius $S'q'KQ'$, and diameter $P'S'p'J$; for all three can in like manner be shown to meet in I upon the directrix. While if focal chords through S' be the ones selected, still more evidently do they, and their director chord, meet the directrix in one point; all three chords being coincident with each other, and with the determining diameter.

Secondly, let two non-focal reciprocal chords be chosen, such as pq and PQ [Figure 3]; or $p'q$ and $P'Q$. Then will they, and their common director chord, EF, or JF, as the case may be, still all three meet the directrix in one point G, or H. For, let chord pq cut the directrix in, say, G', and PQ cut it in G; then G and G' are identical. Join points pf, qf, PS, QS, $G'f$, and GS; and let the corresponding focal chords through p and P cut the directrix, as we have just seen, in one and the same point g.

Then, by a well known theorem, $G'f$ externally bisects the focal angle pfq of chord pqG'; and similarly, GS externally bisects angle PSQ. But these angles pfq and PSQ are equal, since $pfS'=PSd$; and $qfS'=QSd$ [Theorem 6]; and thus the halves of their supplements, $G'fg$ and GSg, are also equal, even as also are gfX and gSX. Therefore the right angled triangles $G'fX$ and GSX are equal in all respects; with $G'X$ equal to GX; and thus G' and G are the same point.

Taking $pPQq$ as a quadrangle, G is thus one of its three summits, S' another summit; and $S'pEP$, $S'qFQ$ harmonic ranges upon it; so that GS', Gqp, GFE, and GQP form a harmonic pencil, with ray GFE passing through the third summit; which is also the intersection of diagonals pQ and Pq. And therefore the two non-focal reciprocal chords pqG and PQG with their director chord EF, all three meet in G upon the directrix.

While, if $p'qH$ and $P'HQ$ be the two reciprocal non-focal chords chosen, with their director chord JFH [Figure 3]; in like manner it can be proved that they all three meet the directrix in H. For this is the meeting point of the external and internal bisectors, Hf and HS, of the focal angles $p'fq$ and $P'SQ$, respectively; since the focal chords through p' and P' meet, as we have seen, in I upon the directrix; and the focal angle $p'fq$ is equal to the supplement of focal angle $P'SQ$; $p'fS'$ being equal to $P'SS'$; and qfS' to QSd [Theorem 6], and thus the right angled triangles HXf and HXS are equal in all respects; and chords $p'qH$ and $P'HQ$ meet in the same point H upon the directrix. Which is also a summit of the re-entrant quadrangle $P'Qqp'$; another summit being S'; $S'qFQ$ and $P'S'p'J$ harmonic ranges upon it [Theorem 12]; and thus HS', Hqp', HFJ, and $P'HQ$ an harmonic pencil. So that the reciprocal non-focal chords $p'qH$ and $P'HQ$, with their common director chord JFH, all three meet in one and the same point H upon the common directrix. While the external intersection of the lines through $P'q$ and $p'Q$, since this is the third summit to the re-entrant quadrangle $P'Qqp'$, must lie on the director chord HFJ, since it is a pencil ray; even as the internal summit h [Theorems 11 and 12. Figure 2] lay upon the director chord and pencil ray EF.

Corollary 1. Conversely, then, chords through any two corresponding

points, such as pP, or $p'P'$ [Figure 3], determined by a common radius $S'pEP$, or diameter $P'S'p'J$; and meeting the directrix in the same point G, or H, etc., are thereby corresponding chords, with two other corresponding points qQ, etc. While EFG, or JFH, etc., will be their director chord, cutting the director circle again in F, etc., the extremity of the common radius $S'qFQ$, or diameter, determining these two new points.

Theorem 15. Of two such reciprocal chords, if one pq', or PP' passes through the center [O or C] of its curve, then the other PQ', or pp', as the case may be, is thereby parallel to the major axis, and thus meets its fellow pq', at M, or m, its perpendicular intersection with the directrix. While $F'EM$, or $E'Em$, the director chord in question, has become collinear with fE and fF', or SE and SE', the two lines joining the extremities of the determining radius and diameter to f or S, the fixed point of that chord which passes through the center [Figure 2].

For, let chord pq' of the ellipse pass through its center O; and chord PMQ' be drawn in the hyperbola parallel to the major axis, thus cutting the directrix perpendicularly in M; and being the directrix distances of P and Q'. Then, both ellipse and hyperbola being symmetrical about their respective major and minor axi; angles $q'fS'=pS'f=PS'S=Q'SS'$; and thus $q'Q'$ are corresponding points, lying on the common determining diameter $Q'S'q'F'$ [Theorem 6]; while pq' and PQ' are reciprocal chords, with M for their meeting point on the directrix [Theorem 14].

And, in like manner, if $PmCP'$ be drawn through the center of the hyperbola, and $p'pm$ be parallel to the major axis of the ellipse, $P'SS'=PS'S=pS'f=p'fS'$; so that pp' is the reciprocal chord to PP', and meets it at m on the directrix, the common foot of the directrix distances pm and $p'm$.

Next, the director chord $F'EM$ is collinear with both fE and fF'. For $pf=pE$, and $S'F'=S'E$; so that fpE and $F'S'E$ are isosceles triangles, with sides $S'E$ and pE collinear, and angles pfE, pEf, and $S'F'E$ equal. And thus their bases $F'E$ and fE are also collinear; and coincident with $F'EM$ the director chord [Theorem 13]. Similarly, if $PmCP'$ and mpp' be the two chords in question, with $E'EM$ for their director chord; SPE and $E'S'E$ are isosceles triangles, with sides PE and $S'E$ collinear; while $E'Em$, being Em produced, must pass through S [Theorem 13]. And thus their bases SE and $E'E$ are also collinear with each other, and with $E'Em$ this director chord.

Corollary 1. Therefore, in such a case, $pfqG$ is parallel to $Q'S'q'F'$, or $PSQG$ to $P'S'p'E'$.

Corollary 2. Obviously, if pP lie on a common radius $S'pEP$, then $q'Q'$ or $p'P'$ must lie on a diameter; and conversely.

Corollary 3. Points p' and q', or P' and Q', are also symmetric, their chords $p'q'$ and $P'Q'$ being perpendicular to the major axis. And thus $P'q'$ and $Q'p'$ are equally inclined to the axi; and likewise $P'S'p'E'$ and $Q'S'q'F'$.

Theorem 16. The normals of corresponding points meet upon the common directrix [Figure 3]. Let pP be the two corresponding points; pr and PR their

respective normals, cutting the directrix in say r and R; $S'pEvP$ their determining radius, meeting the directrix in v; and finally pT and PT their respective tangents, meeting each other in T upon the directrix [Theorem 10]. Let $pfp''g$ and $PSP''g$ be the focal chords through p and P, meeting each other in g upon the directrix [Theorem 14], and determining the other two corresponding points p'' and P'' [Theorem 9].

Then, since pr and pT internally and externally bisect the focal angle $S'pf$, and thus also vpg; therefore $rv:rg=Tv:Tg$ [Euclid VI, 3 and A]. And similarly, $S'P$S, or vPg is internally and externally bisected by PT and PR; so that $Rv:Rg=Tv:Tg$. Therefore, since g, T, and v are identical points in both harmonic ranges, points r and R are also identical.

Theorem 17. The normals of the corresponding focal chords, pp'' and PP'' respectively, meet upon the radius passing through T, the common intersection of their tangents with the directrix, and bisecting their common director chord [Theorem 10, Corollary 2].

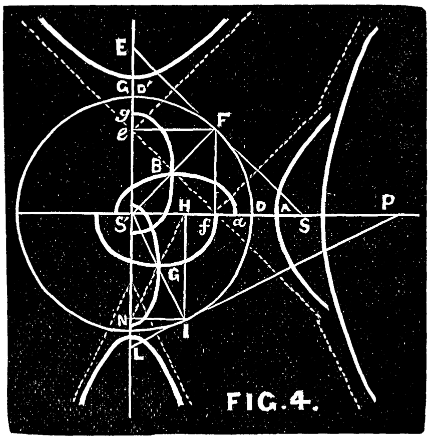


FIG. 4.

For, if such radius, say $S'r'n'TN'R'$ be drawn, meeting the focal chords pp'' and PP'' through p and P in n' and N' respectively; and the normals of p and P in r' and R'. Then, since pr', and pT, PT, and PR', internally and externally bisect the focal angles $S'pf$ and $S'PS$, or $S'pn'$ and $S'PN'$; therefore $S'T:Tn'=r'S':r'n'$; and $S'T:TN'=R'S':R'N'$. While, if the normals of p'' and P'' cut the radius $S'r'n'TN'R'$ in, say, r'' and R'' respectively; then similarly $S'T:Tn'=r''S':r''n'$, and $S'T:TN'=R''S:R''N'$. And S', T, n' and N' being identical points upon the four harmonic ranges $S'r'n'T$, $S'r''n'T$, $S'TN'R'$, and $S'TN'R''$; therefore r' and r'' are also identical points; and likewise R' and R''.

And thus the normals of pp'' meet in r'; and those of PP'' in R', upon the radius $S'r'n'TN'R'$ through T, the common intersection of their four tangents with the directrix; which radius also bisects the common director chord of the reciprocal focal chords.

The foregoing theorems will thus enable us to determine the *Reciprocal* and *Anacyclic* forms to any given conic curve. For the passage of our fixed point, from being the center of its director circle, to a position at infinity, external to said circle, as the focus of a degenerated right line hyperbola, this process, I say, can be now conceived as part of a simultaneous four-fold movement. Since at any point in this course—say, for example, at f [Figure 4]—its generated curve, in this case an ellipse, is balanced and answered by a reciprocal curve, in this case a hyperbola; reverse and opposite in every corresponding point, and gener-

ated by a second fixed point [*e. g.* S], harmonic to the first; rushing from the polar of the center at infinity to meet it on the circumference; and sinking ultimately into the center from whence the original fixed point arose, as that point passes at last to the infinity from whence sprang its fellow. The medial and pivotal conic form in this cycle, as previously stated, being the parabola.

But in addition to these two greater reciprocals, there are also generated two *anacyclic* curves, if we may so name them, or minor reciprocals, in the elliptic and hyperbolic cycles, respectively.

For just as the sine and tangent of the same point [*e. g.* F, or I, Figure 4] on our director circle gave us, on the major axis, the harmonic fixed points of a reciprocal ellipse and hyperbola; so also will the cosine and cotangent of this same point further determine, on the minor axis, two more harmonic fixed points, which will generate a second pair of reciprocal curves—ellipse and hyperbola—anacyclic, and opposite, in many ways, each to its fellow of like form in the first pair. In the hyperbola this anacyclic curve will be a conjugate hyperbola; conjugate, however, not with respect to its position, or magnitude; but solely in regards to its eccentric ratio, or shape. For the common radius [*e. g.* $S'F$, or $S'I$], whose tangent and cotangent pass through the two fixed points generating the said anacyclic hyperbolas, is thereby parallel to an asymptote of each [Theorem 4]; and the said asymptotes, being thus themselves parallel, and so inclusive of supplementary angles with their fellows, must belong to hyperbolas conjugate in ratio. Nevertheless, since the asymptotes are not coincident, but lie on opposite sides of the common determining radius, their respective hyperbolas are not conjugates with respect to position. Nor, again, are they with reference to their magnitude; except, indeed, they be equilaterals; since having a common director circle, their major axi are equal.

The hyperbolic anacyclics are thus conjugates in ratio. And although the same does not hold true of their reciprocal anacyclic ellipses, since the conjugate to an ellipse is never other than itself; yet nevertheless, as the reciprocals of conjugates, they will hold most important relations to each other.

Thus, for example, the right line fe, or HK, joining the anacyclic fixed points, will intersect their common determining radius $S'F$, or $S'I$, on a common extremity, B, or b, of the respective minor axi. Since, joining the fixed points, it is necessarily the bisecting and bisected diagonal to the parallelogram under the sine and cosine of F, or I; and its center B, or b, is thus also the center for the concyclic points $FfS'e$, or $IHS'K$, as the case may be; and therefore, again, is a common extremity of the minor axi [Theorem 3].

The interfocal distance $S'f$, or $S'K$, must therefore ever equal the minor axis of its anacyclic ellipse; since points S' and B, or b, are common; while if C and C', or C'' and C''', represent the respective centers of the two anacyclic curves; $S'C=BC'$, and $S'C'=BC$, or $S'C''=bC'''$, and $S'C'''=bC''$.

And so again; if X be taken to represent the magnitude $S'F$, or $S'D$, the radius of the common director circle, equalling the major axis; and Y be taken to represent the interfocal magnitude $S'f$ of the given ellipse; then the eccentric

ratios of the said ellipse, and of its reciprocal hyperbola will be represented by $Y:X$, and $X:Y$ respectively [Theorem 7]. While the eccentricity of the anacyclic hyperbola, as a conjugate in ratio, will obviously be $X:\sqrt{(X^2-Y^2)}$; and thus the ratio of its reciprocal ellipse—the anacyclic to the first ellipse—will be $\sqrt{(X^2-Y^2)}:X$.

As the reciprocal fixed points then, say P and H [Figure 4], of ellipse and hyperbola move from the center and infinity respectively, towards the circumference and each other, along the major axis, their anacyclic fixed points, L and K, are moving simultaneously in a reverse direction along the minor axis, away from the circumference and each other, towards the center or infinity. Instantaneously transposing with each other, and returning from thence back to the circumference; as the two parent points meet and cross on the circumference and pass onwards towards infinity and the center. The common determining radius—as *e. g.* $S'I$—during these cycles, swinging from the minor to the major axis, and back again. The diagonal—as *e. g.* HK—and the tangent and cotangent—as *e. g.* PIL—meanwhile rocking from a position, on the minor axis, at right angles to each other, through the medial parallels, corresponding to eBf and EFS, to a second position at right angles on the major axis; and then, reversing, back to their original position.

The medial forms, therefore, of both the hyperbolic and the elliptic cycles —lying midway between the parabola, the pivotal form of the greater reciprocal cycle, on the one hand; and its extremes, the circle and right line hyperbola at infinity, on the other—are alike determined by that radius $S'F$, which bisects in 45° the right angle between the axi. Giving us, then, outside of the director circle, two equilateral hyperbolas. And within it, two precisely similar ellipses, whose eccentric ratios, like those of their reciprocal equilateral hyperbolas, will be as $\sqrt{1}$ to $\sqrt{2}$, or $\sqrt{2}:\sqrt{1}$; so that the duplicate ratio of CS to CX, or CX to CS, in both alike, is as 1 to 2.

This medial ellipse, as we may call it, in default of a better term, has many important theorems in common with its reciprocal, the equilateral hyperbola; despite the fact that the elliptic, answering to the latter curve, is usually thought to be the circle. Yet a lack of space compels me to defer to a future occasion the discussion of some of these theorems; as well as others pertaining in general to "reciprocals" and "anacyclics."

USES OF THE SPECIAL TRIPLE PRODUCT $\mathbf{ab}^2$ OF EXTENSIVE QUANTITIES.

By J. V. COLLINS, Stevens Point, Wisconsin.

1. The study of the special triple product of extensive quantities where the second factor is repeated gives interesting results. Thus by means of this product the nature of the quaternion itself can be developed directly from the fundamental laws of multiplication. The author of the *Ausdehnungslehre* developed his subject by using the fundamental laws of operation as a starting point, but writers on quaternions pursue a widely different course.

2. Following out the idea of developing quaternions from the fundamental laws assumed to hold *for products of the reference units only*,* the writer used the special triple product $\mathbf{aB}^2$ where $\mathbf{a}$ and $\mathbf{B}$ are extensive quantities, $\mathbf{B}$ being a unit quantity, rather than the dual product $\mathbf{ab}$, or the more general triple product $\mathbf{abc}$. By Hamilton's law† $\mathbf{aB}^2=-\mathbf{a}$. Then $(\mathbf{aB})\mathbf{B}$ by the associative law should likewise equal $-\mathbf{a}$. Expressing both $\mathbf{a}$ and $\mathbf{B}$ as sums of components along three rectangular reference axes, we may write

$$\mathbf{a}=a_i\mathbf{i}+a_j\mathbf{j}+a_k\mathbf{k},$$

$$\mathbf{B}=B_i\mathbf{i}+B_j\mathbf{j}+B_k\mathbf{k}.$$

Then if M_i, M_j, M_k are the respective minors of m_i, m_j, m_k in the determinant whose respective rows are m_i, m_j, m_k, a_i, a_j, a_k, B_i, B_j, B_k,

$$(\mathbf{aB})\mathbf{B}=-\mathbf{a}=-(a_iB_i+a_jB_j+a_kB_k)\mathbf{B}+\begin{vmatrix} M_j & M_k \\ B_j & B_k \end{vmatrix}\mathbf{i}+\begin{vmatrix} M_k & M_i \\ B_k & B_i \end{vmatrix}\mathbf{j}+\begin{vmatrix} M_i & M_j \\ B_i & B_j \end{vmatrix}\mathbf{k}.$$

For, one has no trouble in interpreting by means of the assumed laws for the products of the units such terms as $(\mathbf{ij})\mathbf{i}$, $(\mathbf{ij})\mathbf{j}$, etc. Then only the three terms which contain the product of all three reference units remain. But after rearranging the terms which contain these products by the commutative and associative laws, and combining, the coefficient of the product vanishes identically. The simplified product just given shows $-\mathbf{a}$ "analyzed" into two components, one along $\mathbf{B}$ and the other perpendicular to it.

If then $(\mathbf{aB})\mathbf{B}=-\mathbf{a}$ we may assume

$$\frac{-\mathbf{a}}{\mathbf{B}}=\mathbf{aB},$$

*See Grassmann's Ausdehnungslehre of 1862, Engel ed., Arts. 37, 42.

†It is easy to prove that if the squares of the reference units equal negative unity, the square of any unit vector has the same value.

which evidently makes of $\mathbf{aB}$ a kind of multiplier or operator that can change $\mathbf{B}$ into $-\mathbf{a}$.

Let us now take the case of three unit quantities, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, of the same kind.

$\mathbf{AB}$ is an operator which changes $\mathbf{B}$ into $-\mathbf{A}$,
$\mathbf{BC}$ is an operator which changes $\mathbf{C}$ into $-\mathbf{B}$,
$\mathbf{CA}$ is an operator which changes $\mathbf{A}$ into $-\mathbf{C}$,
$\mathbf{BA}$ is an operator which changes $\mathbf{A}$ into $-\mathbf{B}$, etc.

Attempting to satisfy these conditions one is led directly to put $\mathbf{AB}=\mathbf{C}$, which by our assumed laws immediately gives

$$\mathbf{BC}=\mathbf{A},\quad \mathbf{CA}=\mathbf{B},\quad \mathbf{CB}=-\mathbf{A},\quad \mathbf{AC}=-\mathbf{B},\quad \mathbf{BA}=-\mathbf{C}.$$

These equations with those already assumed, viz., $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-1$, are the fundamental laws of quaternions. From them, as Tait states,* the whole subject can be developed. We have already seen that $\mathbf{aB}$ is an operator which changes $\mathbf{B}$ into $-\mathbf{a}$: an easy extension interprets the meaning of $(\mathbf{aB})\mathbf{c}$.

Thus starting from the fundamental laws of multiplication only, one is led naturally and directly to the fully developed subject of quaternions or to Macfarlane's *Algebra of Physics.* In this way unity in the form of presentation is secured for all branches of vector analysis. Moreover there is nothing in the preceding to limit the meaning to vectors. Thus $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{a}$, $\mathbf{b}$ may stand for points, lines, and planes and the results be interpreted. This method of developing the subject seems much preferable to the one followed by writers on quaternions who begin arbitrarily with either the "$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$" system of vectors, or with the conception of a quaternion as an operator which changes one vestor into another, these assumptions or definitions being followed later by an investigation into the fundamental laws of multiplication. Moreover this plan has the additional merit of showing just why Hamilton made the square of a unit vector negative rather than positive unity. For had the scalar law chosen been $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=1$, $(\mathbf{aB})\mathbf{B}$ would have come out $-\mathbf{a}$ instead of $\mathbf{a}$ as called for by the associative law. Thus the associative law has to be given up in the algebra of physics (unless the factor i is introduced) and one must write $(\mathbf{Ba})\mathbf{B}=\mathbf{a}$ instead of $(\mathbf{aB})\mathbf{B}=\mathbf{a}$. It should be added however that this does not deprive the algebra of physics of usefulness, for in some ways it is more practical than the quaternions it so closelv resembles.

3. We have just seen to what use the product $\mathbf{aB}^2$ can be put when its parts are used together. Let us next study to what use they can be put when employed independently.

Let us denote extensive quantities by clarendon or black faced type, unit quantities by capitals, tensors by italics, and a perpendiculaı to a vector in positive direction of angle (in plane of factors) by writing a line over it. Then (§2) let

*Tait, Quaternions, Art. 71, 2nd ed.

$\mathbf{a}\,{}^{o}\,\mathbf{b}$=cosine part of product$-(\mathbf{ab})\mathbf{B}=ab.\cos\angle\,\mathbf{ab}\cdot\mathbf{B}$,

$\mathbf{a}\,{}^{i}\,\mathbf{b}$=sine part of product$-(\mathbf{ab})\mathbf{B}=ab.\sin\angle\,\mathbf{ab}\cdot\bar{B}$,

$\mathbf{a}\,{}^{t}\,\mathbf{b}$=tangent function of $\mathbf{a}$ and $\mathbf{b}=ab.\tan\angle\,\mathbf{ab}\cdot\bar{B}$,

and $\mathbf{a}\,{}^{c}\,\mathbf{b}$=cotangent function, $\mathbf{a}\,{}^{z}\,\mathbf{b}$=secant function, and $\mathbf{a}^{k}\,\mathbf{b}$=cosecant function of vectors $\mathbf{a}$ and $\mathbf{b}$. Evidently $\mathbf{a}\,{}^{c}\,\mathbf{b}$, $\mathbf{a}\,{}^{z}\,\mathbf{b}$ differ only by a numerical factor from $\mathbf{a}\,{}^{o}\,\mathbf{b}$, both being measured in same direction as $\mathbf{a}\,{}^{o}\,\mathbf{b}$, while $\mathbf{a}\,{}^{t}\,\mathbf{b}$ and $\mathbf{a}^{k}\,\mathbf{b}$ differ in the same way from $\mathbf{a}\,{}^{i}\,\mathbf{b}$ and are measured in the same direction as $\mathbf{a}\,{}^{i}\,\mathbf{b}$.

4. By means of the expressions of the last articles one can write vector equations in a form corresponding to the cartesian ones, as $\mathbf{r}=\mathbf{x}+\mathbf{y}$, or $\mathbf{r}=\mathbf{x}+\mathbf{y}+\mathbf{z}$. Thus

$$\mathbf{r}=\mathbf{Q}\,{}^{o}\,\mathbf{a}+\mathbf{Q}\,{}^{o}\,\mathbf{b}$$

is the equation of an ellipse where $\mathbf{Q}$ is an auxiliary vector making with $\mathbf{a}$ the so-called eccentric angle, and $\mathbf{a}$ and $\mathbf{b}$ are the semi-axes. Evidently $\mathbf{Q}$ can be understood, whereupon the equation reduces to

$$\mathbf{r}={}^{o}\,\mathbf{a}+{}^{o}\,\mathbf{b}.$$

Similarly $\mathbf{r}={}^{z}\,\mathbf{a}-{}^{c}\,\mathbf{b}$ is the equation of the hyperbola referred to its axes, $\mathbf{r}={}^{c}\,{}^{2}\mathbf{a}-{}^{c}_{\circ}\mathbf{a}$ that of the parabola, the dot under $\mathbf{c}$ denoting that the vector length is measured in the complementary direction to that of $\mathbf{c}$ unmarked, here along $\mathbf{b}$. The same notation is available for higher plane curves. Thus

$\mathbf{r}={}^{z}\,\mathbf{a}\,{}^{z}\,\mathbf{b}$ corresponds to $x^2(y^2-b^2)=a^2y^2$,

$\mathbf{r}={}^{t}\,\mathbf{p}+{}^{o}\,\mathbf{m}+\mathbf{p}+{}^{i}\,\mathbf{m}$ corresponds to $m^2y^2=(p-y)^2(x^2+y^2)$.

In space we have* $\mathbf{r}={}^{o}\,\mathbf{a}+{}^{o}\,\mathbf{b}+{}^{o}\,\mathbf{c}$, as the vector equation of the ellipsoid, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vector axes, and the auxiliary vector $\mathbf{Q}$, understood, is not written. Similarly

$\mathbf{r}={}^{z}\,\mathbf{A}(\mathbf{a}+{}^{o}\,\mathbf{b}+{}^{o}\,\mathbf{c})$ is the equation of the hyperboloid of two sheets,

$\mathbf{r}={}^{k}\,\mathbf{C}[\,{}^{k}\,\mathbf{C}(\,{}^{o}\,\mathbf{a}+{}^{o}_{\bullet}\,\mathbf{b})-{}^{o}\,\mathbf{c}]$ is the equation of the hyperboloid of one sheet,

$\mathbf{r}={}^{c}\,{}^{2}\mathbf{c}+{}^{c}\,\mathbf{c}.\,{}^{z}\,\mathbf{a}(\,{}^{z}\,\mathbf{C}\cdot\mathbf{a}+{}^{o}\,\mathbf{b})$ is the equation of the hyperbolic paraboloid,

$\mathbf{r}=\mathbf{q}\,{}^{o}\,\mathbf{C}+\mathbf{q}\,{}^{c}\,\mathbf{C}(\mathbf{Q}\,{}^{o}\,\mathbf{a}+\mathbf{Q}\,{}^{o}\,\mathbf{b})$ is the equation of the elliptical cone,

where $\mathbf{q}$ is an auxiliary vector whose tensor is equal to that of $\mathbf{r}$, and $\mathbf{a}$ and $\mathbf{b}$ are the axes of the ellipse at a unit's distance from vertex. $\mathbf{r}=\mathbf{q}\,{}^{o}\,\mathbf{C}+\mathbf{q}\,{}^{k}\,\mathbf{C}(\mathbf{Q}\,{}^{o}\,\mathbf{a}+\mathbf{Q}\,{}^{o}\,\mathbf{b})$ is the equation of the elliptical cylinder. To get the normal to a central quadric, say the ellipsoid, one writes down the scalar product of the dyadic

*The derivation of this and the following equations is omitted for want of space.

$$\frac{\mathbf{A}\mathbf{A}'}{a^2}+\frac{\mathbf{B}\mathbf{B}'}{b^2}+\frac{\mathbf{C}\mathbf{C}'}{c^2}$$

and the radius vector $\mathrm{o}\,\mathbf{a}+\mathrm{o}\,\mathbf{b}+\mathrm{o}\,\mathbf{c}$ which gives $\dfrac{\mathrm{o}\,\mathbf{A}}{a}+\dfrac{\mathrm{o}\,\mathbf{B}}{b}+\dfrac{\mathrm{o}\,\mathbf{C}}{c}$. Having the normal it is easy to construct the equation of the tangent plane.

It may be remarked that if the preceding equations were written in one of the usual notations, as, for instance, that used by Gibbs, they would become very complicated, though the concepts represented, as we have seen are simple enough.

5. Neither the usual vector equations, those just given, nor the cartesian equations, possess exclusive merits over the others. Each in turn can be superior to the others for certain purposes. It is easier to construct the curves and surfaces from the equations just given than from the cartesian equations since one can use the trigonometric tables. The equations for the *plane* curves do not differ except in compactness and convenience of notation from the vector equations commonly given. But those for space of three dimensions are new forms. Writers on vector analysis use the ϕ function for quadrics, which introduces very compact expressions, but ones essentially scalar. The use of the above equations makes the treatment of solid space analogous to that of plane and connects vector analysis much more closely to cartesian analysis.

NOTE ON THE TOTIENT OF A NUMBER.

By G. A. MILLER.

Like the preceding paper, "On the Totitives of Different Orders," MONTHLY, Vol. XI, p. 129, this note aims to point out the advantages of employing the theory of groups in the proofs of some of the elementary theorems of number theory.

The number of natural numbers which do not exceed the positive integer m and are prime to m is denoted by $\phi(m)$ according to Gauss.* This function of m is known by various names. English, French, and German writers respectively employ the following names: totient of m, indicator of m, Euler's ϕ-function of m. If $1, d_1, d_2, \ldots\ldots, d_\lambda, m$ are all the divisors of m it is well known that

$$\phi(1)+\phi(d_1)+\phi(d_2)+\ldots\ldots+\phi(d_\lambda)+\phi(m)=m.$$

The main object of this note is to show the connection between this formula and the theory of cyclic groups in a very elementary manner. Several closely related questions in number theory are also treated from the standpoint of group theory.

*Gauss, *Disquisitiones Arithmeticae*, Art. 38.

If d is any divisor of m then there is one and only one subgroup of order d in the cyclic group (G) of order m. This subgroup is cyclic. Hence there are just $\phi(d)$ operators of order d in G. Moreover, the order of every operator of G is a divisor of m. The given formula therefore says nothing more than that *the sum of the numbers of the operators of the different possible orders is equal to the order of G*. That is, this formula exhibits a very elementary property of cyclic groups and requires no further proof if the given properties of cyclic groups are assumed to be known. We proceed to give a simple proof of these properties.

Since d is a divisor of m any generator (s) of G must contain a cyclic subgroup of order d. It cannot contain more than one such subgroup because the 1st, 2d, ..., mth powers of s contain only d operators whose orders divide d,* and these powers give all the operators of G. These statements prove that G contains one and only one subgroup of order d and that this subgroup is cyclic.

The given formula may clearly be regarded as a special case of the following evident theorem. *If g is the order of any finite group and if $1, d_1, d_2, \ldots, d_\gamma$ are the orders of all of its cyclic subgroups, then*

$$\phi(1)+\phi(d_1)+\phi(d_2)+\ldots\ldots+\phi(d_\gamma)=g.$$

Since every non-cyclic group contains more than one cyclic subgroup of the same order, it follows that the given $\gamma+1$ divisors of g cannot all be different unless the group is cyclic. In this special case we arrive at the preceding formula, as has been proved above.

The given properties of the cyclic group may be employed to obtain very elementary proofs of the formulas $\phi(mn)=\phi(m)\phi(n)$ whenever m and n are prime to each other, and

$$\phi(m)=m(1-\frac{1}{p_1})(-\frac{1}{p_2})\ldots\ldots(1-\frac{1}{p_\rho})$$

where $p_1, p_2, \ldots\ldots, p_\rho$ are all prime factors of m.

The former of these two formulas gives the number of operators of highest order in the cyclic group of order mn, m and n being relatively prime. This cyclic group contains just one cyclic subgroup of each of the orders m and n, and it is the direct product of these two cyclic subgroups since they have only the identity in common. To obtain an operator of order mn it is necessary to multiply an operator of order m from the first subgroup into an operator of order n from the second subgroup, and all such products are of order mn. Hence the number of operators of highest order in the entire group is the product of the numbers of the operators of highest order from the two subgroups. In other words, $\phi(mn)=\phi(m)\phi(n)$.

All the subgroups of a cyclic group of order p^α, p being a prime, are con-

*From $(s^x)^d=1$ it follows that $xd=km$, or $x=km/d$. Hence x has one of the values m/d, $2m/d$, ... dm/d mod$m$.

tained in its subgroup of order p^{a-1}. That is, $\phi(p^a)=p^a - p^{a-1}=p^a(1-1/p)$. If $m=p_1^{a_1}p_2^{a_2}\ldots\ldots p_\rho^{a_\rho}$, it follows from this and the preceding formula that

$$\phi(m)=\phi(p_1^{a_1})\phi(p_2^{a_2})\ldots\ldots\phi(p_\rho^{a_\rho})$$

$$=p_1^{a_1}p_2^{a_2}\ldots\ldots p_\rho^{a_\rho}(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots\ldots(1-\frac{1}{p_\rho})$$

$$=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots\ldots(1-\frac{1}{p_\rho}).$$

It may be observed that the formula $\phi(p^a)=p^{a-1}(p-1)$ may be regarded as a special case of the formula $\phi(m^r)=m^{r-1}\phi(m)$, since $\phi(p)$ is evidently equal to $p-1$.

AN APPLICATION OF STIRLING'S INTERPOLATION FORMULA

By FRANK ELMORE ROSS.

Let write the values of $\sin x$ for $x=0$, π, etc., and form the series of differences, as in the ordinary process of interpolation. The result is as follows, Δ_n denoting the nth difference:

x	$\sin x$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	....
0	0						
		+1					
$\frac{1}{2}\pi$	+1		−2				
		−1		+2			
π	0		0		0		
		−1		+2		−4	
$3\pi/2$	−1		+2		−4		....
		+1		−2		+4	
0	0		0		0		....
		+1		−2		+4	
$\frac{1}{2}\pi$	+1		−2		+4		
		−1		+2			
π	0		0				
		−1					
$3\pi/2$	−1						

This scheme extends infinitely in both directions, the difference Δ_n forming a divergent series. It is found however that a convergent series for the

function thus tabulated can be constructed from the above series of differences, by making use of Stirling's interpolation formula.

$$f(a+nw)=f(a)+n\Delta_1+\frac{n^2}{2!}\Delta_2+\frac{n(n^2-1)}{3!}\Delta_3+\frac{n^2(n^1-1)}{4!}\Delta_4$$
$$+\frac{n(n^2-1)(n^2-2^2)}{5!}\Delta_5 \ldots\ldots\ldots \quad (1).$$

In this formula Δ_n stands for the differences which lie in the same horizontal line $f(a)$, in the case of the even differences, and in the case of the odd differences, for the mean of those differences which lie just above and below this line. If we apply (1) to the above scheme, we get at once

$$\sin x = x\left[1-\frac{2(x^2-1)}{3!}+\frac{2^2(x^2-1)(x^2-2^2)}{5!}-\frac{2^3(x^2-1)(x^2-2^2)(x^2-3^2)}{7!}+\ldots\right.$$
$$\left.\ldots\ldots\ldots+\frac{(-2)^n(x^2-1)\ldots\ldots(x^2-n^2)}{(2n+1)!}-\ldots\ldots\ldots\right]\ldots\ldots\ldots \quad (2),$$

in which x is expressed in terms of π. This series is convergent for all values of x, since the ratio of two successive terms

$$\frac{x^2-n^2}{n(2n+1)} \doteq \tfrac{1}{2} \quad (n \doteq \infty).$$

While this series is not of any practical use for computing this function, it is nevertheless believed to be new. From it we can readily obtain a simple series for π, as follows:

Since $\dfrac{\sin x}{x} \doteq \tfrac{1}{2}\pi$, $(x \doteq 0)$, we get by putting $x=0$ in (2),

$$\tfrac{1}{2}\pi = 1+\frac{2}{3!}+\frac{2^2 2^2}{5!}+\frac{2^3 2^2 2^3}{7!}+\ldots\ldots\ldots+\frac{2^n 2^2 3^2 \ldots\ldots\ldots n^2}{(2n+1)!}\ldots\ldots\ldots, \qquad \ldots\ldots\ldots(3)$$

which is convergent, the ratio of convergence being the same as in (2). This series is also believed to be new. It is found to be quite convenient for arithmetical computation. By means of it the value of π correct to eight decimal places was computed in half an hour. There are however better series for this purpose.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

No. 212 was also solved by Jeannette Brooks. Two solutions of No. 213 were received from G. B. Hudson.

215. Proposed by EDWIN L. RICH, Lehigh University.

Solve (1).........$x/a+y/b+c/z=3$,
(2).........$x/a+b/y+z/c=3$,
(3).........$a/x+y/b+z/c=3$.

I. Solution by J. SCHEFFER, Hagerstown, Md.

Subtracting (3) from (1) and (3) from (2) the results may easily be put in the form

$$(z/c)^2-(x/a-a/x)(z/c)=1 \text{.........}(4),$$
$$(y/b)^2-(x/a-a/x)(y/b)=1 \text{.........}(5),$$

whence $z/c=x/a,\ -a/x$; $y/b=x/a,\ -a/x$.

By substituting the first pair of values in (1),

$$x=a,\ y=b,\ z=c, \text{ and } x=\tfrac{1}{2}a,\ y=\tfrac{1}{2}b,\ z=\tfrac{1}{2}c.$$

By substituting the second pair of values we obtain.

$$x=3a,\ y=-b/3,\ z=3c;$$
$$x=-a/3,\ y=3b,\ z=3c,$$
$$x=3a,\ y=3b,\ z=-c/3.$$

II. Solution by L. S. SHIVELY, Mt. Morris, Ill., and ELMER SCHUYLER, Brooklyn High School, New York.

Let $x/a=A$, $y/b=B$, and $z/c=C$. Then the original equations become

$$A+B+1/C=3 \text{.........}(1),$$
$$A+1/B+C=3 \text{.........}(2),$$
$$1/A+B+C=3 \text{.........}(3).$$

Subtracting (2) from (1) and $B-1/B=C-1/C$, hence $B=C,\ -1/C$.

It can similarly be shown that

$$A=B=C \text{.........}(4), \text{ and that}$$
$$A=-1/B;\ B=-1/C;\ C=-1/A \text{.........}(5).$$

From (4) and (1), $2A+1/A=3$. The roots of this quadratic are $\frac{1}{2}$ and 1.
$\therefore A=B=C=\frac{1}{2}$, and $x=\frac{1}{2}a$, $y=\frac{1}{2}b$, $z=\frac{1}{2}c$; $A=B=C=1$, and $x=a$, $y=b$, $z=c$.

From (5) and (1) it is seen that $A=3$, $B=-\frac{1}{3}$, $C=3$.

$\therefore x=3a$, $y=-\frac{1}{3}b$, $z=3C$.

In like manner, $x=3a$, $y=3b$, $z=-\frac{1}{3}c$, and $x=-\frac{1}{3}a$, $y=3b$, $z=3c$.

Also solved by M. E. Graber, Grace M. Bareis, E. L. Sherwood, Christian Hornung, F. P. Matz, G. B. M. Zerr, and the Proposer.

216. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Express by radicals the roots of $x^6+ax^4+bx^3+\frac{1}{4}a^2x^2+\frac{1}{2}abx+c=0$.

I. Solution by E. L. SHERWOOD.

$x^6+ax^4+\frac{1}{4}a^2x^2+bx^3+\frac{1}{2}abx+c=0$, $(x^3+\frac{1}{2}ax)^2+b(x^3+\frac{1}{2}ax)+c=0$, whence we have, by solving the quadratic

$$x^3+\tfrac{1}{2}ax+\frac{2b-b^2+4c}{4}=0, \text{ or } x^3+\tfrac{1}{2}ax+\frac{2b+b^2-4c}{4}=0,$$

whence by Cardan's method, Burnside and Panton, p. 108,

$$x=\sqrt[3]{p}+\frac{-H}{\sqrt[3]{p}},\quad \omega\sqrt[3]{p}-\frac{H}{\omega\sqrt[3]{p}},\quad \omega^2\sqrt[3]{p}-\frac{H}{\omega^2\sqrt{p}}$$

where $p=\frac{1}{2}[\sqrt{(G^2+4H^3)}-G]$, and $G=\dfrac{2b-b^2+4c}{4}$ or $\dfrac{2b+b^2-4c}{4}$, $H=\frac{1}{6}a$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Write the equation as follows:

$$x^2(x^2+\tfrac{1}{2}a)^2+bx(x^2+\tfrac{1}{2}a)+c=0.$$

Let $x(x^2+\frac{1}{2}a)=x^3+\frac{1}{2}ax=y$. $\quad\therefore y^2+by+c=0$.

$\therefore y=\dfrac{-b\pm\sqrt{(b^2-4c)}}{2}$.

Let ω be an imaginary cube root of unity, and let m, n be the roots of $t^2-yt-a^3/216=0$.

$\therefore x=m+n$, $x=\omega m+\omega^2 n$, $x=\omega^2 m+\omega n$.

As y has two values, the six values of x are expressed as radicals.

Also solved by J. Scheffer, G. W. Greenwood, Elmer Schuyler, F. P. Matz, and the Proposer.

217. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the condition that $E\equiv x^5-bx^3+cx^2+dx-e$ shall be the product of a complete square and a complete cube.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The factors must be of the form

$$(x^2-2ax+a^2)(x^3+2ax^2+\frac{4a^2x}{3}+\frac{8a^3}{27}),$$

hence $b=\frac{5}{3}a^2$, $c=-\frac{10}{27}a^3$, $d=\frac{20}{27}a^4$, $e=\frac{8}{27}a^5$, and therefore

$$\sqrt{\frac{3b}{5}}=-3\sqrt[3]{\frac{c}{10}}=\sqrt[4]{\frac{27d}{20}}=\sqrt[5]{\frac{27e}{8}}.$$

II. Solution by E. L. RICH, Lehigh University.

Consider the cube, $x^3+3px^2+3p^2x+x^3$, and the square, $x^2+2qx+q^2$. Their product is,

$$x^5+(2q+3p)x^4+(q^2+3p^2+6pq)x^3+(3pq^2+6p^2q+p^3)x^2+(2p^3q+3p^2q^2)x+p^3q^2.$$

Then the first condition is $2q+3p=0$ or $q=-\frac{3}{2}p$. The other conditions for the values of the coefficients gotten by equating coefficients, and substituting the first condition are,

$$b=\tfrac{15}{4}p^2,\ c=-\tfrac{5}{4}p^3,\ d=\tfrac{15}{4}p^4,\ e=-\tfrac{9}{4}p^5.$$

Also solved by J. Scheffer, G. B. M. Zerr, Elmer Schuyler, and F. P. Matz.

218. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

Prove that $\sum\limits_{r=0}^{n}\frac{c_r}{r+1}=\frac{2^{n+1}-1}{n+1}$ where c_r is the coefficient of x^r in the expansion of $(1+x)^n$.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), and HOWARD M. ARMSTRONG.

$$\sum_{r=0}^{r=n}\frac{c_r}{r+1}=\int_0^1[c_0+c_1x+c_2x^2+\dots\dots\dots+c_nx^n]dx=\int_0^1(1+x)^ndx=\frac{2^{n+1}-1}{n+1}.$$

II. Solution by J. SCHEFFER, Kee Mar College.

By the binomial theorem

$$2^{n+1}-1=(1+1)^{n+1}-1=(n+1)+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\dots\dots\dots$$

$$\therefore\ \frac{2^{n+1}-1}{n+1}=1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\dots\dots\dots=1+\tfrac{1}{2}\binom{n}{1}+\tfrac{1}{3}.\binom{n}{2}+\dots\dots=\sum_{r=0}^{n}\frac{c_r}{n+1}.$$

III. Solution by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

$$\sum_{r=0}^{n}\frac{c_r}{r+1}=c_0+\tfrac{1}{2}c_1+\tfrac{1}{3}c_2+\tfrac{1}{4}c_3+\dots\dots\dots\frac{1}{r+1}+c_r\dots\dots+\frac{1}{n+1}c_n$$

$$=1=\tfrac{1}{2}n+\tfrac{1}{3}.\frac{n(n-1)}{2\,!}+\tfrac{1}{4}.\frac{n(n-1)(n-2)}{3\,!}+\dots\dots\dots\frac{1}{n+1}$$

$$=\frac{1}{n+1}(n+1+\tfrac{1}{2}n(n+1)+\tfrac{1}{3}.\frac{n(n-1)(n+1)}{2\,!}+\dots\dots\dots+1)$$

$$=\frac{1}{n+1}(1+n+1+\frac{n(n+1)}{2!}+\frac{n(n+1)(n-1)}{3!}+\text{.........}+1)-1$$

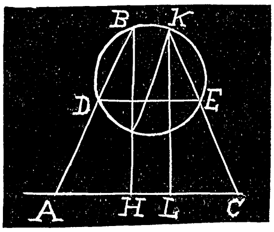
$$=\frac{1}{n+1}(1+1)^{n+1}-1=\frac{2^{n+1}-1}{n+1}.$$

Demonstrated by M. E. Graber by the method of mathematical induction, and by Elmer Schuyler by means of the integral calculus. Also solved by G. B. M. Zerr.

GEOMETRY.

236. Proposed by J. R. HITT, San Marcos, Texas.

If two sides of a triangle pass through fixed points, the third side touches a fixed circle.



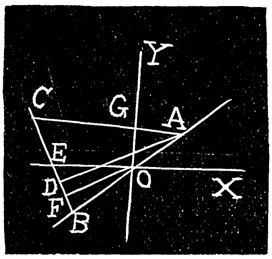
I. Solution by the PROPOSER.

Let the sides AB, BC of triangle ABC pass through the points D, E, respectively. Draw BH perpendicular to AC, from F, intersection of circumcircle of triangle DBE with BH, draw diameter FK. Draw BK, KL perpendicular to AC. Arc DBE is locus of B. Since $\angle ABH$ is constant, F is a fixed point. $\therefore K$ is fixed. FBK is a right angle. $\therefore BHLK$ is a rectangle. $\therefore KL=BH$, a constant. Hence AC touches the circle with center K and radius KL.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O, G be the two fixed points; a, b, c the sides of the triangle; $OG=h$, $OA=r$, $\angle AOX=\angle FOB=\theta$, $AD=p$. Then $AD:OE=c:c-r$.

$\therefore OE=\dfrac{p}{c}(c-r)$.



Also $\cos EOB=p/c$, or $EOB=\cos^{-1}(p/c)=\beta$.

$\therefore \rho\cos(\phi+\beta-\pi-\theta)=\dfrac{p}{c}(c-r)$ is the equation to CB. Let $\phi+\beta-\pi=\delta$.

$\therefore \rho\cos\delta\cos\theta+\rho\sin\delta\sin\theta=\dfrac{p}{c}(c-r)$.

$AG=\sqrt{(h^2+r^2-2rh\sin\theta)}$.

$\therefore h^2=h^2+r^2-rh\sin\theta+r^2-2r\sqrt{(h^2+r^2-2rh\sin\theta)}\cos A$.

$\therefore r^2\sin^2A-2rh\sin\theta\sin^2A=h^2\cos^2A-h^2\sin^2\theta$.

$$\therefore \sin\theta=\frac{1}{h}[r\sin^2A\pm\cos A\sqrt{(h^2-r^2\sin^2A)}],$$

$$\cos\theta=\frac{\sin A}{h}[r\cos A\mp\sqrt{(h^2-r^2\sin^2A)}].$$

$$\therefore \frac{\rho r}{h}\sin A\cos(\delta-A)+\frac{pr}{c}-p=\pm\frac{\rho}{h}\sqrt{(h^2-r^2\sin^2A)}\sin(\delta-A).$$

Let $\frac{\rho}{h}\sin A\cos(\delta-A)+p/c=D$, $\frac{\rho}{h}\sin(\delta-A)=E$.

$\therefore rD-p=\pm E\sqrt{(h^2-r^2\sin^2 A)}$.

$(D^2+E^2\sin^2 A)r^2-2Drp=E^2h^2-p^2$.

$\therefore r=\dfrac{Dp}{D^2+E^2\sin^2 A}$, by differentiation.

$\therefore D^2h^2+E^2h^2\sin^2 A=p^2\sin^2 A$.

Substituting D and E and reducing $\rho^2c^2\sin^2 A+2\rho chp\sin A\cos(\delta-A)$ $=p^2c^2\sin^2 A-p^2h^2$.

$\therefore \rho^2+2\rho H\cos(\delta-A)=G$ is the equation sought, and represents a circle.

Also solved by F. P. Matz.

243. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

What is the equation to the curve on which lie the centers of the inscribed circles in the right-angled triangles of hypotenuse h?

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Take as axes the hypotenuse and its perpendicular bisector. Denoting the length of the hypotenuse by h, and one acute angle by α, the equations to the bisectors of the acute angles may be written

$$y=(x+\tfrac{1}{2}h)\tan\tfrac{1}{2}\alpha,$$
$$y=-(x-\tfrac{1}{2}h)\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha),$$
$$=-(x-\tfrac{1}{2}h)\frac{1-\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\alpha}.$$

By eliminating α between these equations we get as the locus of their intersection, that is, of the center of the inscribed circle, $4(x^2+hy+y^2)=h^2$.

II. Solution by L. S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

It is evident that the angle subtended by the hypotenuse and with vertex at the center of the inscribed circle equals 135°. Since it is constant and equal to 135°, its vertex, and hence the center of the circle, lies upon that arc of a circle constructed upon h as a chord, to contain an angle of 135°. Construct such a circle and join O, its center, with A and B, the ends of h. Then $\angle AOB=90°$. Also $OA=\frac{1}{2}h\sqrt{2}$. Referred to O as origin of coördinates, the equation of the circle is $2x^2+2y^2=h^2$.

Also solved by J. Scheffer, E. L. Sherwood, G. B. M. Zerr, and Elmer Schuyler.

244. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Upon the sides of a triangle as bases isosceles triangles with base angles of 30° are constructed. Show that the lines joining the vertices of these isosceles triangles form an equilateral triangle.

I. Solution by H. M. ARMSTRONG, Cooch's Bridge, Delaware.

Let the sides of the given triangle be denoted by the vectors 2α, 2β, and $2(\alpha+\beta)$. Let i denote a unit vector perpendicular to the plane of the triangle. Then

$$\rho_1=\alpha+\frac{1}{\sqrt{3}}i\alpha \ldots\ldots (1),\quad \rho_2=2\alpha+\beta+\frac{1}{\sqrt{3}}ei\beta \ldots\ldots (2),\quad \rho_3=\alpha+\beta-ei(\alpha+\beta)\ldots\ldots (3)$$

are the vectors drawn to the vertices of the three isosceles triangles.

Hence the vectors of the sides of the new triangle are obviously $\rho_2-\rho_1$, $\rho_3-\rho_2$, $\rho_1-\rho_2$. By means of (1), (2), and (3), we find that $(\rho_2-\rho_1)^2=(\rho_3-\rho_2)^2=(\rho_1-\rho_2)^2$.

Hence $T(\rho_2-\rho_1)=T(\rho_3-\rho_2)=T(\rho_1-\rho_2)$. Therefore the triangle is equilateral.

II. Solution by G. W. GREENWOOD, M. A. (Oxon). Lebanon, Ill.

Denote the vertices by A, B, C, and the remaining vertices of the triangles adjacent to BC, CA, AB, by A', B', C', respectively. I will assume that the triangles are all exterior to the given triangle. Then

$$A'C=a/\sqrt{3},\quad B'C=b/\sqrt{3},\quad \angle A'CB'=60^\circ+C.$$
$$\begin{aligned} A'B'^2 &=\tfrac{1}{3}[a^2+b^2-2ab\cos(60+C)] \\ &=\tfrac{1}{3}[a^2+b^2-ab(\cos C-\sqrt{3}\sin C)] \\ &=\tfrac{1}{6}[a^2+b^2+c^2+2\sqrt{3}\ ab\sin C] \\ &=\tfrac{1}{6}[a^2+b^2+c^2+\sqrt{3}\ abc/r], \end{aligned}$$

where r is the radius of the circumcircle of the triangle ABC. The result shows that $A'B'=B'C'=C'A'$. The same result will follow if A' and A are on the same side of BC, etc.

Also solved by M. E. Graber, J. Scheffer, and G. B. M. Zerr. G. I. Hopkins refers to solution of Problem 96, Vol. 5, No. 4.

245. Proposed by J. H. M. MACLAGAN-WEDDERBURN, M. A., Chicago, Ill.

Given two lines of fixed length, $AB=a$, $BC=b$, perpendicular to each other. A line CP is drawn making $\angle BPC=\theta$. AD is the perpendicular to AB meeting CP in D. Find by Euclidean construction the angle θ such that $AD^2\cos^2\theta+b^2\sin^2\theta$ is a minimum.

Solution by the PROPOSER.

Draw any line CP. From B drop the perpendicular BL upon CP; draw AM parallel to PC meeting BL in M. Join CM. Then $CM^2=CL^2+LM^2=b^2\sin^2\theta+AD^2\cos^2\theta$. The locus of M is a circle described on AB as a diameter with O, the middle point of AB, as center. In order that CM^2, and therefore CM shall be a minimum, the line CM must pass through O.

The construction is therefore the following: Bisect AB in O; produce BOA to P so that $OP=OC$. Then BPC=required angle θ. [Newton's *Principia*, Vol. 2, Section 7, Proposition XXXIV, Scholium.]

245a. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

PCP', DCD' are conjugate diameters of an ellipse; PN, DM are the ordinates to the major axis at P and D; show $CM/PN=CN/DM=AC/BC$, and that AP and BD' are parallel, and that AP' is parallel to BD.

I. Solution by J. SCHEFFER, Kee Mar College, Hagerstown. Md.

Denote AC by a, BC by b, CP by a', CD by b', $\angle PCA$ by ϕ, and $\angle DCM$ by λ. From the well known relations of conjugate diameters of an ellipse, we have

$$\sin^2\phi=\frac{b^2(a^2-a_1^2)}{a_1^2(a^2-b^2)},\ \cos^2\phi=\frac{a^2(a_1^2-b^2)}{a_1^2(a^2-b^2)},$$

$$\sin^2\lambda=\frac{b^2(a^2-b_1^2)}{b_1^2(a^2-b^2)},\ \cos^2\lambda=\frac{a^2(b_1^2-b^2)}{b_1^2(a^2-b^2)}.$$

$$\therefore CM=b_1\cos\lambda=a\sqrt{\frac{b_1^2-b^2}{a^2-b^2}},\ CN=a_1\cos\phi=a\sqrt{\frac{a_1^2-b^2}{a^2-b^2}},$$

$$PN=a_1\sin\phi=b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}},\ DM=b_1\sin\lambda=b\sqrt{\frac{a^2-b_1^2}{a^2-b^2}}.$$

$$\therefore \frac{CM}{PN}=\frac{a}{b}\sqrt{\frac{b_1^2-b^2}{a^2-a_1^2}}=\frac{a}{b},\ \text{since } a_1^2+b_1^2=a^2+b^2,\ \text{and}$$

$$\frac{CN}{DM}=\frac{a}{b}\sqrt{\frac{a_1^2-b^2}{a^2-b_1^2}}=\frac{a}{b}.$$

The coördinates of the point P are $\left(a\sqrt{\frac{a_1^2-b^2}{a^2-b^2}},\ b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}}\right)$, and of A, $(a, 0)$; those of B are $(0, b)$, and of D' $\left(a\sqrt{\frac{b_1^2-b^2}{a^2-b^2}},\ -b\sqrt{\frac{a^2-b_1^2}{a^2-b^2}}\right)$; therefore

$$\text{slope of } AP=b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}}\div a\left(\sqrt{\frac{a_1^2-b^2}{a^2-b^2}}-1\right)=-\frac{b}{a}\frac{\sqrt{(a_1^2-b^2)}+\sqrt{(a^2-b^2)}}{\sqrt{(a^2-a_1^2)}}.$$

$$\text{Slope of } BD'=-\frac{b}{a}.\frac{1+\sqrt{\left(\frac{a^2-b_1^2}{a^2-b^2}\right)}}{\sqrt{\left(\frac{b_1^2-b^2}{a^2-b^2}\right)}}=-\frac{b}{a}.\frac{\sqrt{(a^2-b_1^2)}+\sqrt{(a^2-b^2)}}{\sqrt{(b_1^2-b)}}.$$

By the relation $a_1^2+b_1^2=a^2+b^2$, both slopes are equal; therefore PA is paralle to BD'. In the same way, AP' is parallel to BD.

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Since P and D are extremities of conjugate diameters, their coördinates may be written $(a\cos\phi,\ b\sin\phi)$, $(a\sin\phi,\ -b\cos\phi)$ where a and b are the semi-axes. We obtain the required result at once by disregarding the sign and considering only the absolute magnitudes of the coördinates. Equations to AP and BD are

$$\frac{x-a\cos\phi}{a-a\cos\phi}=\frac{y-b\sin\phi}{-b\sin\phi},\ \text{and}\ \frac{x-a\sin\phi}{-a\sin\phi}=\frac{y+b\cos\phi}{b+b\cos\phi},$$

and the inclinations are therefore

$$-\frac{b\sin\phi}{a(1-\cos\phi)}\ \text{and}\ -\frac{b(1+\cos\phi)}{a\sin\phi},$$

which are easily seen to be equal.

Also solved by G. B. M. Zerr.

CALCULUS.

187. Proposed by L. T. JACKSON, St. Louis, Mo.

Find the area of the ellipse

$$x=a_1+a_2\cos\theta+a_3\sin\theta,$$
$$y=b_1+b_2\cos\theta+b_3\sin\theta.$$

I. Solution by H. B. LEONARD, B. S., Chicago, Ill.

Let $x'=x-a_1$, $y'=y-b_1$; then eliminating $\cos\theta$ and $\sin\theta$ in turn we obtain

$$b_2x'-a_2y'=\triangle\sin\theta,\ b_3x'-a_3y'=-\triangle\cos\theta,$$

where $\triangle=b_2a_3-a_2b_3$. The substitution $x''=b_2x'-a_2y'$, $y''=b_3x'-a_3y'$, which multiplies areas by $\triangle$, transforms the ellipse into the circle $x''=\triangle\sin\theta$, $y''=-\triangle\cos\theta$; *i. e.* $x''^2+y''^2=\triangle^2$.

The area of circle being $\pi\triangle^2$, the area of ellipse is $\pi\triangle=\pi(b_2a_3-a_2b_3)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Transform the origin to the center by writing x for $x-a_1$, y for $y-b_1$.
$\therefore x=a_2\cos\theta+a_3\sin\theta$, $y=b_2\cos\theta+b_3\sin\theta$.

$$\text{Area}=\int y\,dx=\int_0^{2\pi}(b_2\cos\theta+b_3\sin\theta)(a_3\cos\theta-a_2\sin\theta)d\theta$$

$$=\int_0^{2\pi}(a_3b_2\cos^2\theta-a_2b_3\sin^2\theta-a_2b_2\sin\theta\cos\theta\pm a_3b_3\sin\theta\cos\theta)d\theta$$

$$=\pi(a_3b_2-a_2b_3).$$

Also solved by F. P. Matz, and the Proposer.

188. Proposed by SAUL EPSTEEN.

Evaluate $I=\int\dfrac{y(y^2+2)dy}{(y^4+3y^2+3)\sqrt{(y^2+1)}}$.

I. Solution by W. W. BEMAN, M. A., Professor of Mathematics in the University of Michigan.

Putting $y^2+1=z^2$ and $z-1/z=u$, the integrand becomes $\dfrac{1}{u^2+3}$; hence integral $=\dfrac{1}{3^{\frac{1}{2}}}\tan^{-1}\dfrac{1}{3^{\frac{1}{2}}}[(1+y^2)^{\frac{1}{2}}-(1+y^2)^{-\frac{1}{2}}]$.

II. Solution by CHRISTIAN HORNUNG, and M. E. GRABER, Tiffin, Ohio.

Let $y^2+1=u^2$, then $I=\int\dfrac{(u^2+1)du}{u^4+u^2+1}=\frac{1}{2}\int\dfrac{du}{u^2+u+1}+\frac{1}{2}\int\dfrac{du}{u^2-u+1}$

$$=\frac{1}{3^{\frac{1}{2}}}\left[\tan^{-1}\frac{2(y^2+1)^{\frac{1}{2}}+1}{3^{\frac{1}{2}}}+\tan^{-1}\frac{2(y^2+1)^{\frac{1}{2}}-1}{3^{\frac{1}{2}}}\right]$$

*** Dr. G. B. M. Zerr obtains the same result and reduces it to the form

$$\frac{1}{\sqrt{3}}\left[\pi+\tan^{-1}-\frac{(3y^2+3)^{\frac{1}{2}}}{y^2}\right].$$

Also solved by J. Scheffer, E. L. Sherwood, H. M. Armstrong, and G. W. Greenwood.
Solved with same result by F. P. Matz.

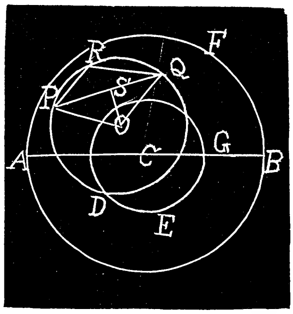
AVERAGE AND PROBABILITY.

130. Proposed by L. C. WALKER, A. M., Santa Barbara, Cal.

Three points are taken at random in a given circle, and a circle passed through them. The probability that the circle through the random points will be wholly in the given circle is $\frac{2}{5}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let PRQ be the triangle formed by joining the three random points P, R, Q in the circle AFB; O the center of the circle circumscribing PRQ. Draw the circle DEG concentric with AFB, both with center C, so that $AD=GB=OP$. Since PQ is known in length and direction, and the angle PRQ is given, if OP is less than AC, the area of DEG represents the number of ways the three points can be taken, so that the circle circumscribing PRQ will lie wholly within AFB.



Let $PQ=2x$, $AC=r$, $\angle POS=\phi$, $\theta=\sin^{-1}(x/r)$, area of segment $PRQ=u$, ρ=angle PQ makes with some fixed line.

Then $PO=x\text{cosec}\phi=r\sin\theta\text{cosec}\phi$, $CD=r-x\text{cosec}\phi=r(1-\sin\theta\text{cosec}\phi)$.

Area $DEG=\pi r^2(1-\sin\theta\operatorname{cosec}\phi)^2$.

$u=x^2(\phi\operatorname{cosec}^2\phi-\cot\phi)=r^2\sin^2\theta(\phi\operatorname{cosec}^2\phi-\cot\phi)$.

An element of surface at Q is $4xdxd\rho=4r^2\sin\theta\cos\theta d\theta d\rho$; at R it is $du=2r^2\sin^2\theta\operatorname{cosec}^2\phi(1-\phi\cot\phi)d\phi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, θ and $\pi-\theta$; of ρ, 0 and 2π.

Hence, doubling as R may be on either side of PQ, we get for the required chance,

$$p=\frac{16\pi r^6}{\pi^3 r^6}\int_0^{\frac{1}{2}\pi}\int_\theta^{\pi-\theta}\int_0^{2\pi}\sin^3\theta\cos\theta\operatorname{cosec}^2\phi(1-\phi\cot\phi)(1-\sin\theta\operatorname{cosec}\phi)^2 d\theta d\phi d\rho$$

$$=\frac{32}{\pi}\int_0^{\frac{1}{2}\pi}\int_\theta^{\pi-\theta}\sin^3\theta\cos\theta\operatorname{cosec}^2\phi(1-\phi\cot\phi)(1-\sin\theta\operatorname{cosec}\phi)^2 d\theta d\phi$$

$$=\frac{2}{3\pi}\int_0^{\frac{1}{2}\pi}[2(\pi-2\theta)\sin2\theta+4-4\cos4\theta-3\sin^2 2\theta\cos2\theta$$

$$+64\sin^4\theta\cos\theta\log\tan\tfrac{1}{2}\theta]d\theta=\tfrac{2}{5}.$$

Also solved by F. P. Matz.

Miscellaneous 146 was also solved by Jeannette Brooks.

PROBLEMS FOR SOLUTION.

ALGEBRA.

223. Proposed by THEODORE L. DELAND, Office of the Secretary of the Treasury, Washington, D.C.

An officer in the Treasury Department assigned three clerks to count a lot of silver dollars and when finished noted that there was an apparent difference in their efficiency; and, to determine the fact, gave to each a similar lot of the same amount to count, the only record made at the time being that A to count his lot alone, took three weeks longer, B took two weeks longer, and C took one week longer than it took for all working together to count the first lot. The best counter, on the record made, was given an efficiency mark of 93 on the scale of 100. What efficiency mark should, on the record, be given to each of the other two counters?

224. Proposed by G. W. GREENWOOD, M. A. (Oxon). Lebanon, Ill.

Show that, if none of the quantities x, y, z is zero, the result of eliminating them from

$$(x+y)(x+z)=bcyz \text{ ....(1)},$$
$$(y+z)(y+x)=cazx \text{ .... (2)},$$
$$(z+x)(z+y)=abxy \text{ .... (3)},$$

is $\begin{vmatrix} \pm a, & 1, & 1 \\ 1, & \pm b, & 1 \\ 1, & 1, & \pm c \end{vmatrix}=0.$

[Oxford, 1896.]

225. Proposed by H. M. ARMSTRONG, Cooch's Bridge, Delaware.

If $\alpha=ax+cy+bz$.........(1), $\beta=cx+by+az$........(2), $\gamma=bx+ay+cz$.........(3), show that $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(a^3+b^2+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

GEOMETRY.

248. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

Given AB, BC in a straight line, to produce it to D so that $AD.CD=BD^2$.

249. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of a square from three of its vertices, to find the side of the square.

250. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of an equilateral triangle from the vertices; to find the side of triangle. [Perkins' *Geometry*, Olney's *Geometry*.]

CALCULUS.

189. Proposed by J. E. SANDERS, Hackney, Ohio.

Solve $d^2y/dx^2=-\beta^2(p+y)$, p and β being constants. The initial conditions are $y=0$ for $x=0$, l; $dy/dx=0$ for $x=l/2$. [Merriman's *Mechanics*, 9th Ed., 1903, §62.]

190. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

$$\int_0^\infty \frac{\sin x \cos\beta x}{x}dx,\ \int_0^\infty \frac{\sin\alpha x\cos x}{x}.$$

MECHANICS.

170. Proposed by M. E. GRÁBER, A. M., Heidelberg University, Tiffin, Ohio.

Prove that the moment of inertia of an ogival head rotating about its geometrical axis is $\frac{\pi w}{g}\int_0^{R\sqrt{(4n-1)}} y^4dx$, where w is the weight in pounds of a cubic foot of material, R the radius of the base of the ogive, and n the diameter of projectile.

DIOPHANTINE ANALYSIS.

123. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Of two numbers $a_ib_ic_id_ie_i$ $(i=1, 2)$ it is given that their 10 digits a_1, ..., e_2 form a permutation of 0, 1, ..., 9, and that the sum of the two is $x3951$. Give an immediate evaluation of x; also list the possible pairs a_1, a_2; .....; e_1, e_2.

AVERAGE AND PROBABILITY.

157. Proposed by J. E. SANDERS, Hackney, Ohio.

A box contains n tickets numbered from 1 to n. How many draws, on the average, will it take to draw all the numbers, each ticket being replaced before drawing again? What is the numerical result for $n=2$ and $n=6$?

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

The University of Missouri has come into possession of the very excellent Loan Exhibit of Mathematical Models which were exhibited at the World's Fair at St. Louis, by Schilling of Halle, Germany. F.

Dr. G. B. Halsted has been made Foreign Associate and Honorary Professor of Mathematics in the University of Tempio, Italy; and also a Fellow of the Royal Astronomical Society. F.

At the last session of the Missouri State Teachers' Association, the mathematics section of that organization resolved to organize into a society to be known as the Missouri Society of Teachers of Mathematics, the aim of the society being the improving of methods and subject-matter of mathematical instruction in the schools of the State, the distributing of literature pertaining to mathematical instruction, and the promoting of social relations between teachers in the secondary schools, graded schools, and colleges of the State. Dr. Hedrick was chosen chairman of the committee on organization, and Dr. Ames chairman of the Society. F.

Professor P. Barbarin, President of the Societé des Sciences Physiques et Naturelles de Bordeaux has acquired the right to translate into French Dr. Halsted's *Rational Geometry* of which he has a long review in the current number of *l'Enseignement Mathematique;* and Yoshio Mikami will translate into Japanese and publish in Japan Dr. Halsted's Vice-Presidential Address to the Association for the Advancement of Science, the subject of which is, The Message of Non-Euclidean Geometry. F.

BOOKS.

The Principles of Mathematical Physics. By Henri Poincairé. Translated by George Bruce Halsted. Pamphlet, 24 pages.

This is a translation of the French savant's address delivered before the International Congress of Arts and Science, at St. Louis, Mo., September, 1904. B. F. F.

Principles of Physics, for Academies, High Schools, and Normal Schools. By Louis Begeman, M. S., Professor of Physics, Iowa State Normal School. 8vo. Cloth, 267 pages. Chicago: Ainsworth & Co.

A splendid work, well suited to elementary instruction. B. F. F.

Exercises in Algebra. By Edward R. Robbins and F. H. Summerville, William Penn Charter School, Philadelphia, Penn. 8vo. Half Leather, 173 pages. New York and Chicago: American Book Co.

This little book will be found helpful to those teachers wishing to supplement the exercises in the text-book they are using. B. F. F.

An Elementary Treatise on the Differential Calculus, founded on the Method of Rates. By William Woolsey Johnson, Professor of Mathematics at the U. S. Naval Academy, Annapolis, Md. Small 8vo. Cloth, xiv+406 pages, 70 figures. Price, $3.00. New York: John Wiley & Sons.

This book is not a revision of the earlier work published by the same author in connection with J. M. Rice, but it is essentially new. The order of subjects differs from the older work, though the method of treatment is very much the same. The author still employs the Newtonian conception of rates, but more closely connected with the method of limits. Each chapter has at the close a fine collection of problems demanding in a variety of ways the applications of the principles therein established. B. F. F.

Statistical Methods with Special Reference to Biological Variation. By C. B. Davenport, Head of the Department of Experimental Biology and Director of Station for Experimental Evolution of the Carnegie Institution. Second Edition, Enlarged. 16mo. Morroco, viii+223 pages. Price, $1.50. New York: John Wiley & Sons.

This edition in addition to the matter contained in the first edition, embodies many of the new statistical methods elaborated chiefly by Prof. Karl Pearsons, his students and associates, and presents a summary of the results gained by these methods. B. F. F.

A Geometrical Political Economy. Being an Elementary Treatise on the Method of Explaining some of the Theories of Pure Economic Science by means of Diagrams. By H. Cunynghame, C. B., M. A., Oxford. At the Clarendon Press, 1904, 128 pp.

This little book is addressed to economists rather than to mathematicians. According to the author (p. 127) "The chief function of Mathematics as applied to Economics is, not to solve problems, but to help us to comprehend truths, which when we have comprehended we may discard the Mathematics as we take down a scaffolding when the building is finished.

Chapter I contains a bibliography of books on this mode of treating Political Economy, supplementing the one in Jevon's *Theory of Political Economy* (3rd ed. 1879).

Chapter II begins with a defense of mathematics against the prejudice which many

students of economics feel. "Mathematical reasoning," says the author, "is only 'ordinary reasoning' assisted by a shorthand mode of expression that enables a proposition to be put in a line and visible in one glance of the eye rather than spread over ten or twelve pages of print. The remainder of the chapter discusses, as the title indicates, Geometrical Diagrams. The fundamental notions of drawing to a scale and change of axes are introduced by means of simple examples, then the terms: axes of coördinates, abscissa, origin, are defined and the methods of adding, subtracting, multiplying and dividing curves are explained and illustrated. Curves are divided into two classes: those which group and present facts, and those which represent laws; the latter are the ones dealt with in this treatise.

In Chapter III, on Demand Curves, money is taken as a standard of value, the reason being that the author desires to have for the simplicity of exposition a unit which may be regarded as invariable. Otherwise, if we consider the demand for (say) linen in terms of cloth, a diminishing utility of linen as it is supplied, involves a changing utility of cloth which its supply brings about (p. 105). In Chapter IV it is shown that a demand curve is a curve of final-utility prices; and one sees easily the characteristic feature of demand curves: that they are *descending curves*, *i. e.* with increasing quantity x, the price y decreases. The product xy denotes the total amount paid for the quantity x, and since people would not give a larger total sum for a smaller quantity of an article than for a larger, the demand curve will cut a rectangular hyperbola xy=constant, negatively (p. 42).

In Chapter V it is shown that Supply Curves are of three types: ascending (agricultural form), constant (industrial form), descending (manufacturing form). The intersection of the demand and supply curves gives the quantity of the article produced and sold.

Chapters VI—X are devoted chiefly to applications of the principles and methods explained above.

In Chapter XI the author explains the "Marshall's Curves," the applications which are made however go beyond those of the inventor. These curves were introduced for the purpose of eliminating the money standard in comparing two articles. Instead of having abscissa and ordinate denote quantity and prices per article, Marshall makes them denote total quantities exchanged. Thus if x and y denote abscissa and ordinate of a point on the price curve, where x=quantity taken, y=price per article, the abscissa and ordinate on a Marshall curve would x and $y=xy$.

On page 83, Fig. 41, two different points are denoted by the letter S. This should be changed to obviate confusion. On page 2 curves are spoken of as those which do and those which do not obey some law. By this the author seems to mean curves for which equations are or are not known. On page 7, it is stated that "mathematics is the science of quantity." In this connection Professor Bocher of Harvard University says:* "the old idea that mathematics is the science of quantity . . . . has pretty well passed away among those mathematicians who have given any thought to the question of what mathematics really is."

On the whole Mr. Cunynghame has carried out creditably his delicate task of writing a book on a subject which is neither Mathematics nor Economics, but is, as the title page states, "an elementary treatise on the method of explaining some of the theories of pure economics by means of diagrams."

SAUL EPSTEEN.

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*The Fundamental Conceptions and Methods of Mathematics. By Maxime Bocher. Address delivered before the Department of Mathematics of the International Congress of Arts and Science, St. Louis, September 20, 1904. Printed in the Bulletin of the American Mathematical Society, Vol. XI, No. 3, Dec. 1904, pp. 115—135.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XII. MARCH, 1905. No. 3.

A GENERAL THEOREM IN LOCAL PROBABILITY.

By Professor ROBERT E. MORITZ, University of Washington, Seattle.

The theorem which forms the objective point of this paper deals with the probability that of n points, distributed at random on a line of length a, the distance between every two shall exceed an arbitrarily assumed distance b. The proof was suggested by the following problem given by Czuber.*

Problem: *In a straight line $AB=a$ two points are assumed at random. It is reqnired to find the probability that their distance apart shall exceed a given length b.*

Czuber gives two solutions which are essentially as follows:

First Solution. The probability that the first point lies within an interval δx_1 is $\delta x_1/a$. The probability that the second point lies within an interval δx_2 is $\delta x_2/a$. The probability that the two points lie simultaneously within these respective intervals is $\delta x_1 \delta x_2/a^2$.

Let each interval be diminished indefinitely, the limiting value $\delta x_1 \delta x_2/a^2$ represents the probability that two points, placed at random on the line AB, are situated at the respective distances x_1 and x_2 from A.

All positions on the line are mutually exclusive and equally possible. The positions favorable to the conditions of our problem are subject to the restriction that the difference between x_1 and $x_2 > b$.

Suppose $x_1 < x_2$, and let x_1 be taken at random. Then every value of x_2 between the limits x_1+b and a gives a favorable case, while x_1 may have any value from 0 to $a-b$. The probability of the favorable cases subject to the condition $x_1 < x_2$ is therefore given by the double integral

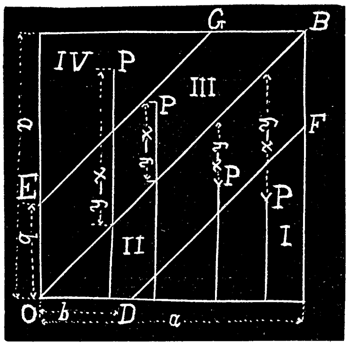
$$\int_0^{a-b}\int_{x_1+b}^{a}\frac{dx_1 dx_2}{a_2}=\frac{1}{a^2}\int_0^{a-b}(a-b-x_1)dx_1=\tfrac{1}{2}\left(\frac{a-b}{a}\right)^2$$

Geometrische Wahrscheinlichkeiten, u. Mittelwerte, §11.

and remembering that for every favorable case $x_1 < x_2$ there is another for which $x_1 > x_2$, we have for the required probability

$$P = \left(\frac{a-b}{a}\right)^2.$$

The second solution is geometrical. As before denote by x_1 and x_2 the distances of two points from A., Next construct a square having the length a for a side, and let P be any point within this square whose coördinates, with reference to a pair of intersecting sides of the square for axes, are x and y.



There is now a one-to-one correspondence between the coördinates x, y and the distances x_1, x_2 of the points on the line from A, to every point in the square there will correspond one and only one point pair on the line and vice versa. The differences between the coördinates x, y correspond to the distances between the prsitions on the line. We can therefore find the required probability by comparing the area of the region of the square for which the difference between x and $y > b$ with the area of the whole square. Take $OD = OE = b$, and draw DF and EG each parallel to the diagonal OB. Denote the parts into which the square is divided by these lines by I, II, III, IV, respectively. Then plainly for every point in

$$\begin{array}{ll} \text{I} & x-y>b, \\ \text{II} & x-y<b, \\ \text{III} & y-x<b, \\ \text{IV} & y-x>b, \end{array}$$

that is, the strips I and IV contain all the points satisfying thr conditions of our problem and no other points.

Now I and II combined constitute a square whose side is $a-b$, and consequently we have for the required probability

$$P = \left(\frac{a-b}{a}\right)^2.$$

Each of these proofs can be readily extended to the case of three points. We will state this case as

Problem 2. *In a straight line $AB = a$ three points are assumed at random. What is the probability that the distance between each pair of points exceeds a given distance b?*

First Solution. Let P_1, P_2, P_3 represent the three points and x_1, x_2, x_3 their respective distances from the point A.

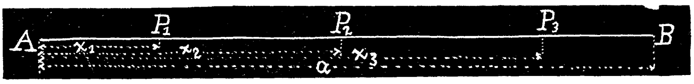
With reference to the positions of the points relative to each other six mutually exclusive cases are equally possible, namely,

$$x_1<x_2<x_3,$$
$$x_1<x_3<x_2,$$
$$x_2<x_3<x_1,$$
$$x_2<x_1<x_3,$$
$$x_3<x_1<x_2,$$
$$x_3<x_2<x_1,$$

consequently the total number of favorable cases is equal to six times the number of favorable cases—subject to one of these conditions, say to $x_1<x_2<x_3$.

Reasoning precisely as in the case of two points we see that the possibility that P_1, P_2, P_3 are at the distances x_1, x_2, x_3 from A is $dx_1dx_2dx_3/a^3$, and that the required probability is the triple integral of this expression, the limits of integration being so chosen as to include all favorable and to exclude all unfavorable positions. For $x_1<x_2<x_3$ these limits are

for P_3 from x_2+b to a,
for P_2 from x_1+b to $a-b$,
for P_1 from 0 to $a-2b$.



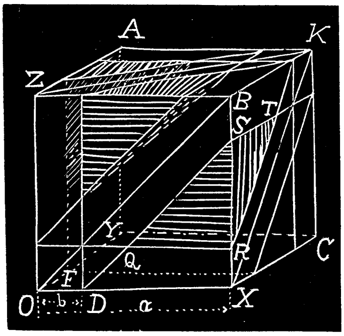
$$\therefore P=\frac{6}{a^3}\int_0^{a-2b}\int_{x_1+b}^{a-b}\int_{x_2+b}^{a}dx_1dx_2dx_3=\int_0^{a-2b}\int_{x_1+b}^{a-b}(a-b-x_2)dx_1dx_2$$
$$=\frac{3}{a^3}\int_0^{a-2b}(a-2b-x_1)^2dx_1=\left(\frac{a-2b}{a}\right)^3.$$

Second Solution. Consider a cube of side a. Take one set of concurrent edges of this cube for axes of coördinates and denote by x, y, z the coördinates of any point P within this cube. To every triplet of points P_1, P_2, P_3 on the line AB corresponds a single point in the cube, viz, the point whose coördinates are $x=x_1$, $y=x_2$, $z=x_3$; to every point in the cube corresponds a single triplet of points on the line, viz, the points whose distances from A are respectively $x_1=x$, $x_2=y$, $x_3=z$. The one-to-one correspondence between the points in the cube and the different portions of three points on the line being established, the problem is solved by comparing the volume of the regions of the cube for which the coördinates of the points satisfy the simultaneous conditions

the difference between x and $y>b$,
the difference between y and $z>b$,
the difference between z and $x>b$,

with the volume of the entire cube.

Take $OD=OE=OF=b$, and through each of the points F and D draw planes parallel to the diagonal plane $OZKC$.



For every point between the planes thus drawn we have the difference between x and $y<b$; for every point in these planes the difference between x and $y=b$; and for the points exterior to the planes we have the difference between x and $y>b$. Again, if planes be drawn through each of the points E and F parallel to the diagonal plane $OXKA$, we see that the difference between y and $z>b$ only for points which lie exterior to both of these planes, and finally in order that the difference between z and $x>b$ the points must be exterior to the planes drawn through E and D respectively, parallel to $OYKB$. The points which satisfy simultaneously the three conditions

the difference between x and $y>b$,
the difference between y and $z>b$,
the difference between z and $x>b$,

are therefore limited to those regions of the cube which lie exterior to each of these three sets of parallel planes, that is to six equal tetrahedrons such as $QRST$.

The dimensions of this tetrahedron are $QR=RS=ST=a-2b$, since $BS=OD=a$ and also $RX=OE=a$, and the volume of $QRST=\frac{1}{6}(a-2b)^3$.

The required probability is therefore

$$P=\frac{6QRST}{\text{volume of entire cube}}=\left(\frac{a-2b}{a}\right)^3.$$

We will now prove the general

Theorem. *The probability that of n points, distributed at random along a line of length a, no two shall fall within the distance b of each other is*

$$\left[\frac{a-(n-1)b}{a}\right]^n.$$

We denote by $x_1, x_2, x_3, \ldots\ldots, x_n$ the distances of the n points from one extremity of the line a, and first compute the probability that the distance between every pair of points exceeds b when the points are subject to the condition

$$x_1<x_2<x_3<\ldots\ldots<x_n.$$

Denote this probability by Q. Then since the suffixes 1, 2, 3, ..., n, may be permuted in $n!$ number of ways, each permutation giving rise to a different distribution of points equally likely, we shall have for the required probability $P=n!Q$. Reasoning exactly as in the case of three points we see that

$$Q=\int_0^{a-(n-1)b}\int_{x_1+b}^{a-(n-2)b}\cdots\cdots\int_{x_{n-2}+b}^{a+b}\int_{x_{n-1}+b}^{a}\frac{dx_1}{a}\frac{dx_2}{a}\cdots\cdots\frac{dx_{n-1}}{a}\frac{dx_n}{a},$$

or if we drop suffixes, since there is no danger of confusion,

$$Q=\int_0^{a-(n-1)b}\int_{x+b}^{a-(n-1)b}\cdots\cdots\int_{x+b}^{a-b}\int_{x+b}^{a}\frac{dx^n}{a^r}.$$

To evaluate this integral we observe that in the evaluation of

$$\int_{x+b}^{a-2b}\int_{x+b}^{a-b}\int_{x+b}^{a}dx^3=\int_{x+b}^{a-2b}\int_{x+b}^{a-b}(a-b-x)dx^2=\int_{x+b}^{a-2b}\tfrac{1}{2}(a-2b-x)^2dx$$

$$=\frac{1}{3!}(a-3b-x)^3,$$

occur the expressions $a-b-x$, $\frac{1}{2}(a-2b-x)^2$, $\frac{1}{3!}(a-3b-x)^3$

Let $u_k=\frac{1}{k!}(a-kb-x)^k$, $u_0=1$. Then

$$\int_{x+b}^{a-kb}u_k dx=\int_{x+b}^{a-kb}\frac{1}{k!}(a-kb-x)^k dx=\frac{1}{(k+1)!}[a-(k+1)b-x]^{k+1}=u_{k+1},$$

and hence we have the chain of expressions,

$$u_{n-1}=\int_{x+b}^{a-(n-2)b}u_{n-2}dx,\quad u_{n-2}=\int_{x+b}^{a-(n-3)b}u_{n-3}dx,\ \cdots\cdots,\ u_1=\int_{x+b}^{a}u_0dx=\int_{x+b}^{a}dx.$$

Eliminating successively the quantities u_{n-2}, u_{n-3}, ..., u_1 on the right, and substituting for u_{n-1} its value, we obtain

$$n_{n-1}=\int_{x+b}^{a-(n-2)b}\cdots\cdots\int_{x+b}^{a-b}\int_{x+b}^{a}dx^n=\frac{1}{(n-1)!}[a-(n-1)b-x]^{n-1},$$

$$Q=\frac{1}{a^n}\int_0^{a-(n-1)b}u_{n-1}dx=\frac{1}{a^n}\int_0^{a-(n-1)b}[a-(n-1)b-x]^{n-1}dx$$

$$=\frac{1}{a^n}\cdot\frac{1}{n!}[a-(n-1)b]^n.$$

$$\therefore P=n!\,Q=\left(\frac{a-(n-1)b}{a}\right)^n.$$

Corollary I. The probability that of n points, distributed at random on a line of length a, at least two shall fall within a distance b of each other is $1-P$.

Corollary II. The probability p that of n points, distributed at random on a closed curve of length a, no two fall within an arc b of each other is

$$p=\left(\frac{a-nb}{a}\right)^{n-1}.$$

For let one point be located at random and let the arc be cut at that point and straightened. No point which falls within a distance b of either end of this line gives a favorable case. We may therefore consider the favorable cases in the distribution of the remaining $n-1$ points on the line of length $a-2b$. The probability therefore involves one less integral sign and the limit of variation is not a but $a-2b$. The denominator however remains a, since this is the measure of the total number of possible cases. Replacing therefore n by $n-1$ in the expression P, and the a in the numerator by $a-2b$ we obtain p.

Corollary III. The probability that of n points distributed at random on a closed curve of length a, at least two fall within a distance b of each other is $1-p$.

NUMERALS FOR SIMPLIFYING ADDITION.

By R. A HARRIS, Washington. D. C.

Primitive numerals are composed of the sign for unity repeated as many times as there are units in the numbers represented. Systems in which all nine figures are primitive were used by the Egyptians in their hieroglyphic writing, the Phœnicians, the Babylonians in their cuneiform inscriptions, the early Romans, and probably the early Indians. By virtue of such a notation, the process of addition can be reduced to counting by ones. The meaning of the symbols being self-evident does not have to be learned.*

In many instances, the first two, three, or four numerals are essentially primitive while the others are more arbitrary or superficial; *e. g.*, the hieratic symbols of the Egyptians, the ordinary Roman numrals, the moderately early Indian numerals, and the numerals now used by the Chinese.

The Palmyrenes made use of two component signs in constructing the first nine numerals,—simple strokes representing the ones and a certain Y-like symbol the component five found in each of the numbers five, six, seven, eight, and nine. Here addition can be performed by counting by ones and by fives,—the fives, for convenience, being taken in pairs so that most of the counting is really by tens instead of by fives. The same is true of the Roman numerals of the republican period where IIII is written for IV and VIIII for IX.

*Columns 3, 5, 6, and 7 of the accompanying figure are copied from the Encyclopedia Britannica, article "Numerals."

The Syrians made use of three component signs; one for unity, another for two, and another for five. Here addition can be performed by counting by ones, twos, and fives (but for the most part, of course, by tens instead of fives).

Although our numerals underwent considerable change in form prior to the invention of the art of printing, there appears to be no doubt that the Europeans received them (zero included) from the Arabs, and that the Arabs received them from the Hindoos. The origin of these symbols is obscure. The most reasonable supposition seems to be that the significant digits (or at least those greater than three) were in the first place the initial letters of the Sanskrit words for the numbers which they represent.

Ingenious attempts have been made to show how each of these nine figures might have originally consisted of as many straight marks, conveniently joined together, as the number itself contains units; or how, instead of lines, angles were the quantities considered. History, however, does not confirm either of these hypotheses, although the first may apply to the figures one, two, and possibly three. But whatever their origin, our figures became, centuries ago, mere signs of numbers conveying no suggestion of decomposition into component parts.*

In the second column of the accompanying table is a proposed set of figures, in which the component ones are vertical strokes, the twos horizontal strokes, and the fives closed loops. These figures are fairly distinctive however turned, can be made with comparative ease, and possess the important property of being obviously composed of ones, twos, and fives, as were the Syrian characters.

The stroke constituting figure two has been slightly curved so as to better distinguish it from the minus sign or the dash, and to prevent its being mistaken for figure one when it happens to be turned into the vertical. The strokes in figure four are likewise curved to distinguish the latter from the sign of equality, and to prevent confusion between the symbols for four and eleven.

The figures one, six, and eight of the proposed system closely resemble their ordinary or Arabic forms.

A convenient rule for adding a column of moderate length may be given thus: Count by twos the horizontal marks (all terminating to the right); upon this sum count the ones, known by the vertical marks; upon this result count by fives the closed loops, generally by tens (the loops being taken in pairs. The result of these three counts is the sum of the column.

To add the next column (to the left), begin by counting by twos if the

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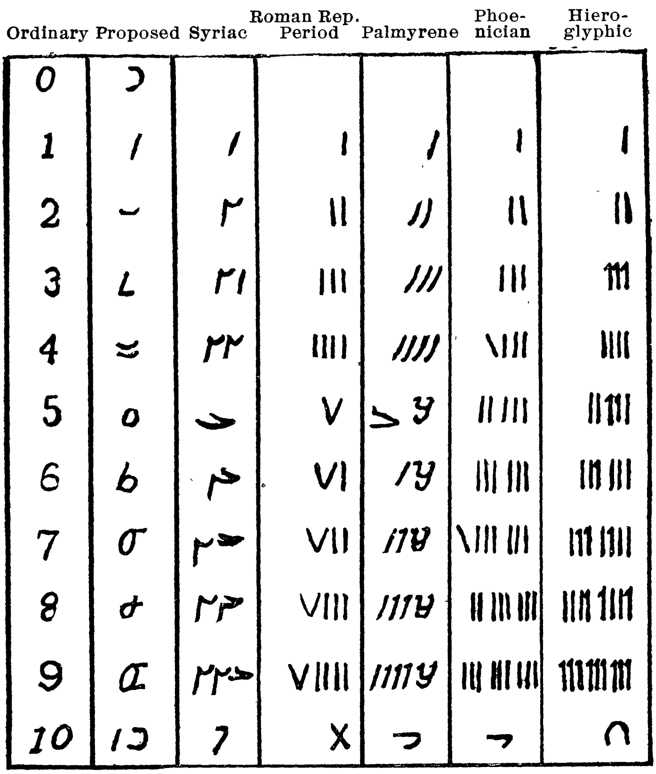
*The following references are to works in which the history of numerals is considered:
Hankel: *Zur Geschichte der Mathematik in Altertum und Mittelalter.*
Brooks: *The Philosophy of Arithmetic*, pp. 21-31, 101-107, 141-146.
Encyclopedia Britannica, under "Numerals" and "Inscriptions."
Cantor: *Vorlesungen ueber Geschichte der Mathematik*, Vol. I, and Plate at end of Vol. I.
Fine: *The Number-System of Algebra*, pp. 79-90.
Ball: *A Short Account of the History of Mathematics*, 2d Ed., pp. 123-131, 141, 158, 161, 189-191.
Fink: *A Brief History of Mathematics*, pp. 6-17.
Encyclopedie des Sciences Mathematiques, I, 1, 1, pp. 20-21.

number carried be even. If it be odd, before proceeding mark a component one of the column to be added by placing a dot above it and count one upon the odd number carried thus making it even. Then proceed as in the first column, but taking care to omit the dotted component one when the ones are counted.

If the person adding can count equally well the odd numbers by twos, or if there should happen to be no component one in the column to be added, the borrowing of a one will not be resorted to.

In the case of a very long column, it may be best to simply count the



ones, twos, and fives (not *by* twos and *by* fives), to then multiply the number of twos by two, the number of fives by five, and finally to add together the three partial sums. This last step is supposed to be done as side work of which only the total sum is to be preserved.

These numerals are proposed for the purpose of saving mental labor in performing addition. The time required in adding by the above methods is about the same as that required by the nsual method, or possibly a little greater, but the mental strain is much less because of the symmetry of the process. The liability to mistakes is somewhat less and for the same reason.

Many subtractions are self-evident; *e. g.*, 87−52=35. Here the compo-

nents of the subtrahend are simply stricken from the minuend, and the result of this cancellation is the difference required, *i. e.*, $87-52=35$.

If characters like those proposed were in use, it seems probable that children could learn addition tables with far greater ease and certainty than is the case at present.

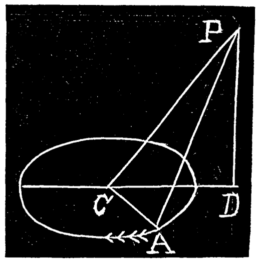
Finally, it may be remarked that if addition be carried out in the ordinary manner, these characters will be quite as serviceable as are the Arabic numerals, and being adapted to the mode of addition described above, a new and convenient way of proving the work will be afforded.

THE CONVEX SURFACE OF AN OBLIQUE CONE.

By F. P. MATZ, Reading, Pa.

Legendre, in his *Théorie des fonctions elliptiques*, has devised a method for finding the convex surface of an oblique cone having a circular base.

We propose the following method: Let $CA=r$, $PD=a$, and $\angle PCD=\omega$. Represent the uniformly varying $\angle DCA$ by θ; then since $CD=a\cot\omega$, we have by trigonometry,



$$\cos\theta=\frac{a^2\cot^2\omega+r^2-(AD)^2}{2ar\cot\omega}\text{.........}(1).$$

$$\therefore AD=\sqrt{(a^2\cot^2\omega+r^2-2ar\cot\omega\cos\theta)}\text{.........}(2),$$

and $AP=\sqrt{(a^2\operatorname{cosec}^2\omega+r^2-2ar\cot\omega\cos\theta)}$ ....(3),

which *line* represents an element of convex surface of the oblique cone.

$$\therefore S=2r\int_0^\pi\sqrt{(a^2\operatorname{cosec}^2\omega+r^2-2ar\cot\omega\cos\theta)}d\theta\text{..........}(4).$$

Transforming (4) under the supposition that $\theta=\pi+2\phi$, we get

$$S=2r\int_0^{\frac{1}{2}\pi}\sqrt{(a^2\operatorname{cosec}^2\omega+r^2+2ar\cot\omega\cos2\phi)}\times2d\phi\text{.........}(5),$$

$$=4r\int_0^{\frac{1}{2}\pi}\sqrt{(a^2\operatorname{cosec}^2\omega+r^2+2ar\cot\omega-2ar\cot\omega\sin^2\phi)}d\phi\text{..........}(6),$$

$$=4r\int_\theta^{\frac{1}{2}\pi}\sqrt{\left[\left(\frac{a^2+2ar\sin\omega\cos\omega+r^2\sin^2\omega}{\sin^2\omega}\right)-\left(\frac{2ar\sin\omega\cos\omega}{\sin^2\omega}\right)\sin^2\phi\right]}d\phi\text{..........}(7).$$

Representing *in order* these parenthetical expressions by m^2 and n^2, (7) gives

$$S=4mr\int_0^{\frac{1}{2}\pi}\sqrt{[1-(n^2/m^2)\sin^2\phi]}d\phi\,.........(8);$$

that is, according to Legendre's system of notation for elliptic functions,

$$n^2/m^2=k^2=\frac{2ar\sin\omega\cos\omega}{a^2+2ar\sin\omega\cos\omega+r^2\sin^2\omega}\,.........(9).$$

By means of (9), we have from (8),

$$S=4mr\int_0^{\frac{1}{2}\pi}\sqrt{(1-k^2\sin^2\phi)}d\phi=4mrE(k,\tfrac{1}{2}\pi)$$

$$=2\pi mr\left[1-\sum_{p=1}^{p=\infty}\left(\frac{1.3.5\,.........\,(2p-1)}{2.4.6\,.................\,2p}\right)^2\left(\frac{k^{2p}}{2p-1}\right)\right]\,.........(10),$$

which is the formula required.

Corollary. If $\omega=90°$, $k^2=0$; and under this supposition, we have from (8),

$$S=4r\sqrt{(r^2+a^2)}\int_0^{\frac{1}{2}\pi}d\phi=2\pi r\sqrt{(r^2+a^2)},$$

which is the formula for the convex surface of a right circular cone.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problem 99. March, 1899; March, 1900; April, 1904.

Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

In the "seven" problem the writer has used the following method to show that there is a single solution, notation apart. Let the natural order *1234567* be contained in every solotion. There are fifteen arrangements containing *124* which do not conflict with the natural order. The same is true for *125*, and for *126*, while there are seventeen for *127*. Advancing the figures by two units, the arrangements containing *346*, *348*, *341*, *342* are obtained; similarly for *561*, etc., then for *713*, etc., ending with the sets containing *672*, *673*, *674*, *675*. In this

way all but twenty-three arrangements out of the complete number of possible arrangements have been written down, and these are also tabulated. Returning to the arrangements in which *34* appears, all which contain *12* are struck out. From those containing *56*, all which contain *12* or *34* are struck out, similarly throughout the other arrangements. There are 225 sets of three arrangements each, in which besides the natural order there may be written an arrangement containing *124* and one containing *125*. But only about one hundred of these are admissible since the others fail owing to conflicting "triads" in the second and third arrangements. These sets of three arrangements are next numbered consecutively. Many of them by simple transformations, in some cases by cyclic changes, are transformed to later ones. To the remaining ones, all non-conflicting arrangements involving *126* are added in turn, and to these very numerous sets now containing four arrangements each, all arrangements of *127* are added in turn. But whenever any two arrangements in a set are capable of being transformed into a later one of the set of one hundred mentioned above, that set is discarded. Thus sets of four, five, and six arrangements are obtained, though by reason of conflicting triads their numbers do not increase as radidly as might be supposed. By taking note of the derivation of the individual arrangements involving *346*, *347*, etc., these transformations are often quickly discovered. The essential feature of the method is the transformation of the uncompleted sets to later sets of the "one hundred."

Some idea of the success with which this was accomplished is gained from the fact that the final solution—for there is apparently only one—was found three times, instead of once; while it might have been found as many times as there are ways of transforming the solution.

Problems 215 and 217 were also solved by L. E. Newcomb, Los Gatos, Cal. No. 117 was also solved by F. P. Matz.

219. Proposed by Dr. SAUL EPSTEEN, The University of Chicago.

Sum to infinity $\frac{1.2}{3}+\frac{2.3}{3^2}+\frac{3.4}{3^3}+\ldots\ldots$

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The series is the expansion of $\frac{2}{3}(1-\frac{1}{3})^{-3}$, and the required sum is therefore $\frac{2}{3}(\frac{2}{3})^{-3}$ or $\frac{9}{4}$.

II. Solution by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

This is a recurring series whose general term is $n(n+1)x^n$. Since the scale of relation is $(1-x)^3$ in which $x=\frac{1}{3}$, we have $(1-x)^3S=2x$.

$\therefore S=2x/(1-x)^3=2\frac{1}{4}$.

Also solved by M. E. Graber, Grace M. Bareis, J. H. Meyer, J. Scheffer, L. E. Dickson, G. B. M. Zerr, and H. Heaton.

220. Proposed by L. ROBINSON, B. S., Philadelphia, Pa.

Find the sum of the first $n+1$ terms of the series

$$1+\frac{m}{1!}+\frac{m(m+1)}{2!}+\frac{m(m+1)(m+2)}{3!}+\text{.........}$$

Solution by HEATON HEATON, Atlantic, Iowa.

Writing $m+1$ for m, the series becomes

$$1+\frac{(m+1)}{1!}+\frac{(m+1)(m+2)}{2!}+\text{.....}+\frac{(m+1)(m+2)\text{.....}(m+n)}{n!}$$

$$=\frac{1}{1!}+\frac{2(m+1)}{2!}+\frac{3(m+1)(m+2)}{3!}+\text{.....}+\frac{n(m+1)(m+2)\text{.....}(m+n-1)}{n!}$$

$$+\frac{(m+1)(m+2)\text{.....}(m+n)}{n!}.$$

Transposing all the terms ot the second number but the last we obtain

$$1+\frac{m}{1!}+\frac{m(m+1)}{2!}+\text{.....}+\frac{m(m+1)(m+n-1)}{n!}=\frac{(m+1)(m+2)\text{.....}(m+n)}{n!}.$$

Also solved by Lloyd Holsinger, J. Scheffer, Grace M. Bareis, G. B. M. Zerr, L E. Dickson, G.W. Greenwood, F. P. Matz.

221. Proposed by F. P. MATZ, Ph. D.. Sc. D.

Eliminate the unknowns from

$x/y+y/z+z/x=a$ ....(1), $x/z+y/x+z/y=b$ ....(2),
$(x/y+y/z)(y/z+z/x)(z/x+x/y)=c$ ....(3).

Solution by GRACE M. BAREIS. A. B , Bala. Pa.

Substituting in (3) the values from (1) it becomes

$(a-z/x)(a-x/y)(a-y/z)=c$, or $a^3-(z/x+x/y+y/z)^2+(z/y+y/x+x/z)a-1=c$,

whence $ab-1=c$ or $ab-c-1=0$.

Also solved by J. Scheffer, Lloyd Holsinger, Henry Heaton, M. E. Graber, L. E. Dickson, G. B. M. Zerr, G W. Greenwood, and the Proposer.

222. Proposed by G. W. WALKER, Camden, N. J.

Extract the square root of $87-12\sqrt{42}$.

Solution by J. H. MEYER. S. J., Spring Hill College, Mobile, Ala.

Let $87-12\sqrt{42}=(\sqrt{\alpha}-\sqrt{\beta})^2$; then $\alpha+\beta=87$; $\alpha\beta=1512$; hence $\alpha=63$, $\beta=24$, and $\sqrt{\alpha}-\sqrt{\beta}=3\sqrt{7}-2\sqrt{6}$.

Also solved by Henry Heaton, Lloyd Holsinger, J. J. Keyes, M. E. Graber, L. E. Dickson, S. F. Norris, J. Scheffer, Grace M. Bareis, G. B. M. Zerr, G. W. Greenwood, L. S. Shively, and F. P. Matz.

GEOMETRY.

247. Proposed by SETH PRATT, C. E., Tecumseh, Neb.

From two given points without a circle to draw two lines meeting in the circumference and making equal angles with the tangent at that point.

Solution by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Let C be the center of the circle, T_1PT_2 the tangent to the circle at P; P_1 and P_2 the given points; $PC=r$, $\angle P_1PT_1=\angle P_2PT_2=\phi$, and the polar coördinates of P_1 and P_2, (ρ_1, θ_1) and (ρ_2, θ_2), respectively.

Then $P_1P_2=\sqrt{[\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)]}$ ....(1).

Also, since $\angle P_1PC=\angle P_2PC=90°+\phi$,

$$P_1P=\sqrt{(\rho_1^2+r^2+2\rho r\sin\phi)}\text{ ....(2), and } P_2P=\sqrt{(\rho_2^2+r^2+2\rho_2 r\sin\phi)}\text{ .... (3).}$$

From the $\triangle P_1PP_2$, $\angle P_1PP_2=180°-2\phi$, we have

$$\cos(180°-2\phi)=-\cos2\phi=\frac{(P_1P)^2+(P_2P)^2-(P_1P_2)^2}{2(P_1P)(P_2P)}\text{ .... (4).}$$

From (4) by means of (1), (2), (3), we have after putting $(\rho_1+\rho_2)=a$, $\rho_1\rho_2=b$, $(\rho_1^2+r^2)=c$, $(\rho_2^2+r^2)=d$, $\rho_1 r=e$, $\rho_2 r=f$, the following equation:

$$16ef\sin^6\phi+8(cf+de)\sin^5\phi+4(cd-4ef)\sin^4\phi-8(cf+de)\sin^3\phi$$
$$-(a^2r^2+4cd+4ef)\sin^2\phi+2[cf+de-ar^3-abr\cos(\theta_1-\theta_2)]\sin\phi$$
$$+[cd-r^4-b^2\cos^2(\theta_1-\theta_2)-2br^2\cos(\theta_1-\theta_2)]=0\text{ ....(5),}$$

for the determination of $\sin\phi$.

Knowing $\sin\phi$, the value of P_1P and P_2P can be determined from (2) and (3).

Also solved by J. Scheffer.

248. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

Given AB, BC in a straight line, to produce it to D so that $AD.CD=BD^2$.

I. Solution by GRACE M. BAREIS, A. B., Bala, Pa.

Describe a circle on AC as diameter, and construct the diameter EF perpendicular to AC. At the point G where EB (or FB) cuts the circle again, draw the tangent. This tangent will cut AC at the required point, D.

For $\angle EGD$ is measured by $\frac{1}{2}(\text{arc}GD+\text{arc}DE)$ and $\angle GBD$ is measured by $\frac{1}{2}(\text{arc}GD+\text{arc}AE)$, since $\text{arc}DE=\text{arc}AE$, $\therefore \triangle GBD$ is isosceles, *i. e.*, $BD=GD$.

But $AD.CD=GD^2$. $\therefore AD.CD=BD^2$.

II. Solution by R. D. CARMICHAEL, Hartselle, Ala., and SAM I. JONES, Gunter, Texas.

Produce AC indefinitely to H. Lay off $BM=AB$. Find the third pro-

portional to CM and BC. From C measure the distance CD=this third proportional. D is the required point.

For $BC^2=CM.CD$, or $BC^2=(AB-BC)CD$. $\therefore CD.AB=BC^2+BC.CD$.

$$CD.AB+BC.CD+CD^2=BC^2+2BC.CD+CD^2.$$
$$CD(AB+BC+CD)=(BC+CD)^2.$$
$$AD.CD=BD^2.$$

Also solved by S. A. Corey, J. R. Hitt, F. D. Posey, M. E. Graber, W. W. Landis, and G. W. Greenwood.

249. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of a square from three of its vertices, to find the side of the square.

250. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of an equilateral triangle from the vertices; to find the side of triangle. [Perkins' *Geometry*, Olney's *Geometry*.]

I. Solution by F. D. POSEY, San Mateo, Calif.

Consider the general case, viz: Given the distances of a point in the plane of a regular n-gon to three consecutive vertices, to find the side of the n-gon.

Let the vertices be A, B, C in order, say clockwise, and P the given point. Let PA, PB, $PC\equiv a$, b, c, respectively. Let $\angle ABP=\alpha$, $\angle PBC=\beta$, taking these clockwise if P be without the angle ABC, and counter-clockwise if P be within. Call the side of the n-gon, x.

There are now two cases: (1) P within the angle ABC of the n-gon, (2) P without this angle. In the first case $\alpha+\beta=\frac{n-2}{n}\pi$. $\therefore \cos\beta=-\cos\frac{2\pi}{n}\cos\alpha+\sin\frac{2\pi}{n}\sin\alpha$. $\therefore \sin\alpha=\operatorname{cosec}\frac{2\pi}{n}\cos\beta+\cot\frac{2\pi}{n}\cos\alpha$.

In the second case $\alpha+\beta=2\pi-\frac{n-2}{n}\pi$. $\therefore \sin\alpha=-\operatorname{cosec}\frac{2\pi}{n}\cos\beta-\cot\frac{2\pi}{n}\cos\alpha$

Now $\cos\alpha=\frac{b^2+x^2-a^2}{2bx}$ (when c is between b and a we have $\cos(2\pi-\alpha)=\cos\alpha$), and $\cos\beta=\frac{b^2+x^2-c^2}{2bx}$ (when a is between b and c we have $\cos(2\pi-\beta)=\cos\beta$). In both cases (1) and (2), $\sin^2\alpha+\cos^2\alpha=(\operatorname{cosec}\frac{2\pi}{n}\cos\beta+\cot\frac{2\pi}{n}\cos\alpha)^2+\cos^2\alpha=1$, which equation after substituting the above values for $\cos\alpha$ and $\cos\beta$ reduces to:

$$\left[\left(\cot\frac{2\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}\right)^2+1\right]x^4+2\left\{\left(\cot\frac{2\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}\right)\left[\cot\frac{2\pi}{n}(b^2-a^2)\right.\right.$$
$$\left.\left.+\operatorname{cosec}\frac{2\pi}{n}(b^2-c^2)\right]-a^2-b^2\right\}x^2+\left[\cot\frac{2\pi}{n}(b^2-a^2)+\operatorname{cosec}\frac{2\pi}{n}(b^2-c^2)\right]^2$$
$$+(b^2-a^2)^2=0.$$

In the case of the square, $n=4$. $\therefore \cot\frac{2\pi}{n}=0$, and $\operatorname{cosec}\frac{2\pi}{n}=1$.

The equation then becomes $2x^4-2(a^2+c^2)x^2+(b^2-a^2)^2+(b^2-c^2)^2=0$.

In the case of the equilateral triangle, $n=3$.

$\therefore \cot\frac{2\pi}{n}=-\frac{1}{\sqrt{3}}$, $\operatorname{cosec}\frac{2\pi}{n}=\frac{2}{\sqrt{3}}$, and we have $x^4-(a^2+b^2+c^2)x^2+a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2=0$.

These equations may be solved as quadratics and by taking a unit length the expression for x may be constructed.

II. Solution by G. I. HOPKINS, Manchester, N. H.

Let AB, CD, and EF be the distances from the three vertices, and H the given vertical angle. Draw $\angle JKL=\angle H$ and make its sides both equal to AB, or any one of the three lengths. Then with J and L as centers, and CD and EF as radii fix the point N. Join NK. Then construct the isosceles $\triangle NKO$, NK being one of the equal sides, and vertical $\angle NKO=\angle H$.

$\therefore NOK$ is the required triangle. For, join JO. $JK=LK$, $NK=OK$, $\angle NKL=\angle JKO$. $\therefore \triangle JOK=\triangle NLK$. $\therefore NL=OJ$. $\therefore \triangle NKO$ is isosceles with vertical $\angle H$, and the three distances JK, JO, and JN are equal, respectively, to AB, CD, and EF.

If the triangle is equilateral, then $\angle H$ is two-thirds of a right angle, and we have the same construction.

Since a diagonal of a square divides it into two isosceles triangles whose vertical angles are 90°, therefore if $\angle H=90°$ we have the same construction and NK is a side of the required square.

Also solved by Henry Heaton, W. W. Landis, J. R. Hitt, M. E. Graber, and G. W. Greenwood.

CALCULUS.

189. Proposed by J. E. SANDERS, Hackney, Ohio.

Solve $d^2y/dx^2=-\beta^2(p+y)$, p and β being constants. The initial conditions are $y=0$ for $x=0$, l; $dy/dx=0$ for $x=l/2$. [Merriman's *Mechanics*, 9th Ed., 1903, § 62.]

I. Solution by W. D. STAYTON, Student, Louisiana State University, Baton Rouge, La.; HOWARD L. STOKER, Lehigh University, and T. O. WHITAKER.

Multiplying by $2dy$ and integrating the resulting equation, we have

$$\left(\frac{dy}{dx}\right)^2=-2\beta^2y-\beta^2y^2+c_1 \text{.........} (1).$$

Since $dy/dx=0$, for $y=\triangle$, $c_1=\beta^2\triangle^2+2\beta^2p\triangle$. Substituting this value of c_1 and extracting the square root, we have

$$dy/dx=\beta\sqrt{(-2py-y^2+\triangle^2+2\triangle p)}=\beta\sqrt{[(\triangle+p)^2-(y+p)^2]}.$$

$$\therefore \beta dx = \frac{dy}{\beta\sqrt{[(\triangle+p)^2-(y+p)^2]}} = \frac{\dfrac{dy}{\triangle+p}}{\sqrt{1-\left(\dfrac{y+p}{\triangle+p}\right)^2}}.$$

$$\therefore \beta x = \sin^{-1}\left(\frac{y+p}{\triangle+p}\right)+c_2 \ldots\ldots\ldots (2).$$

Since $y=0$, for $x=l$, $c_2=\beta l-\sin^{-1}\left(\dfrac{p}{\triangle+p}\right)$.

$$\therefore \beta x = \sin^{-1}\left(\frac{y+p}{\triangle+p}\right)+\beta l - \sin^{-1}\left(\frac{p}{\triangle+p}\right) \ldots\ldots\ldots (3).$$

Since $y=\triangle$ for $x=\frac{1}{2}l$, we have by substituting in (3) and solving for $\triangle$,

$$\triangle = p\left(\sec\frac{\beta l}{2}-1\right).$$

Substituting this value of $\triangle$ in (3), and reducing, we have

$$y+p=p\left(\frac{\sin\beta(l-x)+\sin\beta x}{\sin\beta l}\right).$$

II. Solution by SAUL EPSTEEN, Ph. D., The University of Chicago, and S. A. COREY, Hiteman, Iowa.

Substituting $z=p+y$, the equation takes the form $d^2z/dx^2=-\beta^2 z$, of which $z=c_1\sin\beta x+c_2\cos\beta x$ (c_1 and c_2 being arbitrary constants) is the most general solution. The initial conditions are now $z=p$ for $x=0$, l; $dz/dx=0$, for $x=\frac{1}{2}l$. From the first, we have $c_2=p$, $c_1=p\tan\beta\frac{1}{2}l$, and therefore

$$y=-p+p\tan\beta\tfrac{1}{2}l\sin\beta x+p\cos\beta x.$$

It can now be verified that the condition $z=0$, for $x=l$ is fulfilled and is therefore superfluous in the enunciation of the problem.

The solution may evidently be transformed to the following form:

$$y=\frac{p(\cos\beta\frac{1}{2}l+\sin\beta\frac{1}{2}l\,\sin\beta x+\cos\beta\frac{1}{2}l\,\cos\beta x)}{\cos\beta\frac{1}{2}l}=\frac{p}{\cos\beta\frac{1}{2}l}[-\cos\beta\tfrac{1}{2}l+\cos\beta(x-\tfrac{1}{2}l)].$$

Also solved by G. W. Greenwood, G. B. M. Zerr, J. O. Mahoney, and W. W. Landis.

190. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

$$\int_0^\infty \frac{\sin x\cos\beta x}{x}dx,\ \int_0^\infty \frac{\sin ax\cos x}{x}.$$

The finite discontinuities occur in both cases for the parameters $\alpha, \beta=\pm 1$.

Also solved by M. E. Graber, G. W. Greenwood, F. D. Posey, S. A. Corey, G. B. M. Zerr, and J. O. Mahoney.

MECHANICS.

170. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Prove that the moment of inertia of an ogival head rotating about its geometrical axis is $\frac{\pi w}{g}\int_0^{R\sqrt{(4n-1)}} y^4 dx$, where w is the weight in pounds of a cubic foot of material, R the radius of the base of the ogive, and n the diameter of projectile.

Solution by LYTLE BROWN, Lehigh University, and the PROPOSER.

An ogival head is one-half the solid generated by the revolution of a segment of a circle about its chord. The equation of the generating curve is $y=\sqrt{(4n^2R^2-x^2)}-(2n-1)R$, the origin being the center of the base of the ogive. If $y=0$, the resulting value of x is the length of the head, that is, the length of the head is $R\sqrt{(4n-1)}$. The moment of inertia of an elementary cylinder about the axis of x is $4\pi y^2\frac{\omega}{g}.y^2dx$, or $dI=\frac{\pi\omega}{2g}y^4dx$.

$$\therefore I=\frac{\pi\omega}{g}\int_0^{R(\sqrt{4n-1})} y^4 dx.$$

AVERAGE AND PROBABILITY.

155. Proposed by E. B. WILSON, Ph. D., Yale University.

The game of craps is played with two dice. If the player throws 7 or 11 on the first throw he wins. If he throws 12, 2, or 3 he loses. If the player throws any other number, that is to say, 4, 5, 6, 8, 9, 10, he is obliged to continue throwing until he throws that number again or until he throws 7. If he succeeds in throwing his first throw before he does 7, he wins—otherwise he loses. Required the odds against him. (Note that he can continue throwing indefinitely without getting either his original throw or the 7).

REMARK. A correct solution of this problem is given by Dr. Zerr, in Vol. X, No. 3, p. 81. Mr. J.E. Sanders sent in a different solution, his answer being $\frac{244}{495}$, in favor of the player.

Problem 131 should be numbered 158. Three solutions of this problem have been received, none of which agree.

Problem 157 should be numbered 159,

MISCELLANEOUS.

144. Proposed by IRA M. DeLONG, Professor of Mathematics in the University of Colorado, Boulder, Col.

Determine the number of distinct spherical polygons of n sides formed by arcs of n given great circles on a sphere, each arc to be greater than 0 degrees, and less than 360 degrees, and no two sides of any polygon to lie an the same circle.

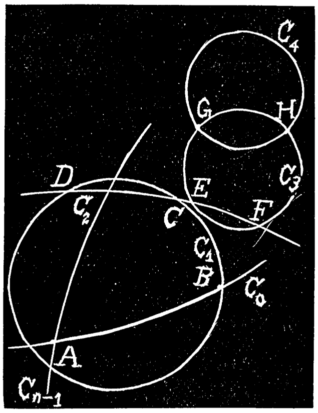
Solution by **ARNOLD EMCH**, Professor of Graphics and Mathematics, University of Colorado.

1. No two sides of any of the required polygons must lie on the same circle. Arcs between 0 and 360° are admitted.

2. Designate the n great circles by $c_0, c_1, c_2, c_3, \ldots\ldots\ldots, c_{n-1}$, Take any of the circles, say c_0, and on it all arcs which may be taken as sides of required polygons. Construct all possible polygons of n sides, x in number, each having one of these arcs as a side.

Since each of all possible polygons of n sides on the sphere has one of its sides on c_0, it is clear that the number of these polygons is identical with x.

There are $2(n-1)$ points of intersection of c_0 with the $(n-1)$ remaining circles. These may be combined in $\{2(n-1)[2(n-1)-1]\}/2$ ways to form arcs of the required polygons. But as no two arcs shall be on one circle we have to exclude from this number those arcs which are formed on c_0 by each of the $(n-1)$ circles separately; *i. e.*, $(n-1)$. The number of arcs on c_0, each of which may be taken as a side of a polygon, is therefore $(n-1)[2(n-1)-1]-(n-1)$, or, since we can also take their supplements to 360°,



$$2(n-1)[2(n-1)-1]-2(n-1)=2^2(n-1)(n-2).$$

4. Take now one of these arcs AB and the two circles c_1 and c_{n-1} through B and A. The question is, in how many different ways can I pass from A to B, from B on c_1 to points of arcs of c_2, $c_3, \ldots\ldots\ldots, c_{n-2}$; then on arcs of these c's to points on c_{n-1} and back to A. For this purpose take any of the permutations of the c's, say $c_2, c_3, c_4, \ldots\ldots\ldots, c_{n-2}$, and let c_1 cut c_2 in C and D, c_2 cut c_3 in E and F, c_3 cut c_4 in G and H, $\ldots\ldots\ldots$, c_{n-2} cut c_{n-1} in X and Y, as indicated in the adjoining diagram.

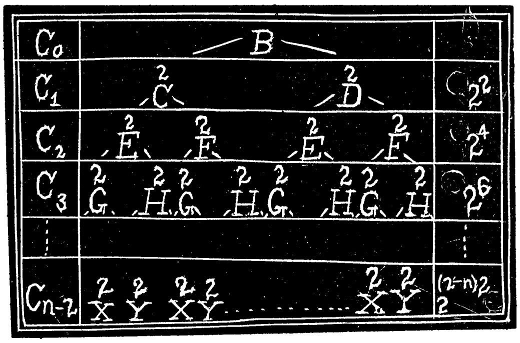
From B, I can reach C and D each in two ways; from C and D, I can, on c_2, reach each of E and F in two ways; so that from B, E, and F may each be reached in 2^3 ways, or both in 2^4 ways. This process continued may be illustrated by the following table.

In this table the first column contains the circles on which the points indicated in the same row are reached. For instance, the points G and H are reached from E and F by describing arcs on the circle c_3. The last column shows in how many ways the points in each row as a whole may be reached from B. Thus all G's and H's in the fourth row may be reached from B in 2^6 different ways. From the table it is seen that the points X and Y which are also on the circle c_{n-1} may be reached from B in 2^{2n-4} different ways. From X and Y each, A may be reached in two ways; hence A may be reached in the prescribed manner in 2^{2n-3} different ways. The same number is obtained for every permutation of $c_2, c_3, \ldots\ldots\ldots, c_{n-2}$; so that there are $(n-3)!2^{2n-3}$ polygons having AB as a side.

5. As there are $2^2(n-1)(n-2)$ different arcs on c_0, the total number x of polygons is

$$2^2(n-1)(n-2)(n-3)!2^{2n-3},$$

$$\text{or } x=(n-1)!2^{2n-1}.$$



6. To find the total number of all possible spherical polygons that may be formed by arcs of n great circles, it is clear that there will be $(r-1)!2^{2r-1}$ polygons of r sides, and consequently

$$\sum_{r=3}^{r=n}[{}^nC_r.(r-1)!2^{2r-1}]$$

will be the number of all possible polygons.

7. *Examples.* With three great circles on a sphere $(3-1)!2^{6-1}=64$ triangles may be formed. The number of spherical triangles and quadrangles which may be formed by four great circles is

$$(4-1)!2^{8-1}+{}^4C_3.(3-1)!2^{6-1}=768+4.64=1024.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

226. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Find the real roots of the system

$$x^2+w^2+v^2=a^2,\quad vw+u(y+z)=bc,$$
$$w^2+y^2+u^2=b^2,\quad wu+v(z+x)=ca,$$
$$v^2+u^2+z^2=c^2,\quad uv+w(x+y)=ab.$$

227. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Solve $x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3=11$; $x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^2y^3+2x^2y^4+4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4+x^2y^5+x^3y^5=30$.

GEOMETRY.

251. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Represent the vertices of any regular polygon by the consecutive numbers 1, 2.....p.....q.....r.....n. To find the sides and area of the triangle formed by joining p, q, and r.

252. Proposed by FREDERICK R. HONEY, Ph. B., Trinity College, Hartford, Conn.

Two plane mirrors form an angle which is less than 45°. Any two points are assumed within this angle in a plane perpendicular to the intersection of the mirrors. A ray of light passes through one point, and after being reflected twice at each mirror, it passes through the second point. Find the path of the ray.

253. Proposed by SAM I. JONES, Gunter Bible College, Gunter, Texas.

The number of cubic inches contained by two equal opposite spherical segments, together with the number of cubic inches contained by the cylinder included between these segments, is 600; if this be $\frac{2}{3}$ of the number of cubic inches contained by the whole sphere, find the height of the cylinder.

CALCULUS.

191. Proposed by J. E. SANDERS, Hackney, Ohio.

A fly goes along a radius of a moving carriage wheel from center to circumference while the wheel makes n revolutions. If each move uniformly, what is the equation to the curve described by the fly in space, and what is its length when the wheel has made $1/m$ of a revolution?

192. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Show that the volume V of the hyper-ellipsoid with semi-axes a_1, a_2, a_3, a_4, etc., in space of $2n$ and $2n+1$ dimensions is

$$V_{2n}=\frac{a_1.a_2.a_3\ldots\ldots a_{2n}.\pi^n}{1.2.3.4.5\ldots\ldots\ldots\ldots n};\qquad V_{2n+1}=\frac{2^{n+1}.a_1.a_2.a_3\ldots\ldots\ldots a_{2n+1}.\pi^n}{1.3.5.7.9\ldots\ldots\ldots\ldots\ldots(2n+1)}.$$

193. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the eccentricity of the maximum semi-ellipse inscribed in a given isosceles triangle.

MECHANICS.

173. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A squirrel is in a cylindrical cage and oscillating with it about its axis which is horizontal. At the instant when he is at the highest point of the oscillation, he leaps to the opposite extremity of the diameter and arrives there at the same instant as the point at which he left. Determine his leap completely.

174. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

By what per cent. is the striking force of a hailstone increased in falling 1000 feet through a stratum of atmosphere moving uniformly eastward at the rate of 60 miles an hour?

DIOPHANTINE ANALYSIS.

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find (1) three square numbers whose sum is a cube; (2) three cube numbers whose sum is a square.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

What *values* of x will make $E=\dfrac{(x+7)\,(x+5)}{(x-7)\,(x-5)}$ represent square numbers?

AVERAGE AND PROBABILITY.

160. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions. Find the mean value of that portion containing the center of gravity.

161. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

A triangle is inscribed at random in a circle; (a) what is the chance the triangle is *oblique;* and (b) what is the chance the triangle is *less in area* than $\frac{1}{4}\pi r^2$?

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

The year 1905 *began*, and will *end*, on a Sunday. Prove that this can not occur again until the year 2015.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If an *unknown* curve be described under a constant acceleration not tending to the center and the hodograph is a cardioid, what is the unknown curve?

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

BOOKS.

Academic Algebra. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in Teachers College, Columbia University, New York. 8vo. Half Leather, 383 pages. Price, $1.25. Boston and Chicago: Ginn & Co.

A splendid work for Academies and High Schools. B. F. F

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APPLICATION OF SEVERAL THEOREMS IN NUMBER THEORY TO GROUP THEORY.

By DR. G. A. MILLER, Stanford University.

Two previous notes (MONTHLY, Vol. 11, p. 129, and Vol. 12, p. 41) were devoted to exhibiting the advantages of employing elementary theorems in group theory in the proof of some of the fundamental theorems of number theory. The present note, on the contrary, aims to translate into the language of group theory several other important theorems of number theory. The object is to make these theorems more available to the student of group theory rather than to emphasize the advantages resulting from the employment of group theory concepts in the study of number theory. As most of the developments relate to ground which is decidedly common to the two subjects it is hoped that some of the results are also useful from the standpoint of number theory.

It is well known that a cyclic substitution of even order generates a group (C) in which half the substitutions are even and the other half are odd. The even substitutions are squares and the odd substitutions are non-squares under C.* As any substitution (s) has the same square as s multiplied by the substitution of order 2 contained in C, it follows that the two square roots of any even substitution are either both odd or both even when the order of C is divisible by 4. When this order is not divisible by 4 one of these square roots is even and the other is odd.

Let C be the group of isomorphisms of the cyclic group of order p, p being any odd prime. With each substitution of C may be associated a number which is congruent (mod p) to the index of the power into which this substitution

*The even substitutions constitute a subgroup whose order is half the order of C.

transforms the operators of the cyclic group of order p. With respect to modulus p these numbers constitute a group which is simply isomorphic with C.* If k is any one of these numbers the corresponding substitution is even or odd as $\left(\frac{k}{p}\right)$ is $+1$ or -1; *i. e.* as k is a quadratic residue or quadratic non-residue of p.† Even substitutions are sometimes called positive and odd substitutions are called negative. If this is done the Legendre symbol $\left(\frac{k}{p}\right)$ exhibits whether the substitution corresponding to k is positive or negative.

The preceding developments remain unchanged if C is regarded as the group of isomorphisms of the cyclic group of order p^α, as should be the case since a quadratic residue of p is also a quadratic residue of p^α and vice versa. That is, if we associate with the substitutions of the cyclic group of order $p^{\alpha-1}(p-1)$ the indices (mod p^α) of the powers into which these substitutions transform the operators of a cyclic group of order p^α, each substitution is positive or negative as $\left(\frac{k}{p^\alpha}\right)$ is $+1$ or -1, k being the corresponding index. From the preceding paragraph it follows that just half of the first $m(p-1)$ natural numbers which are prime to p are quadratic residues of p while the other half are quadratic non-residues, m being arbitrary. From the present paragraph it is clear that just half of the first $p^{\alpha-1}(p-1)$ numbers which are prime to p are quadratic residues of p^α. As a quadratic residue of p^α is also a quadratic residue of p, the last two sentences furnish a direct proof of the theorem that a quadratic residue of p is also a quadratic residue of p^α.‡

The main results of the last two paragraphs may also be stated as follows: *If the $p^{\alpha-1}(p-1)$ numbers which are less than p^α and prime to p are arranged in any order whatsoever and are then multiplied by any number (k) prime to p, the products (modulo p^α) give a rearrangement of these numbers which represents a substitution of degree $p^{\alpha-1}(p-1)$, whenever k is not identical to 1 mod p^α. This substitution is positive or negative as $\left(\frac{k}{p}\right)$ is $+1$ or -1.* For the case when $\alpha=1$ this theorem was proved by Zolotareff and is of fundamental importance in his proof of the law of quadratic reciprocity by means of substitution theory.§

If p and q are distinct odd primes, the cyclic group (C) of order $p^\alpha q^\beta$ may be represented as an intransitive substitution group of degree $p^\alpha + q^\beta$ and this is the smallest number of letters on which it can be represented. Suppose that C is represented in this way. The group of isomorphisms (I) of C is the direct product of the two cyclic groups of orders $p^{\alpha-1}(p-1)$, $q^{\beta-1}(q-1)$, respectively, and may be represented as an intransitive substitution group involving two systems of intransitivity of degrees $p^{\alpha-1}(p-1)$ and $q^{\beta-1}(q-1)$, respectively.

**Annals of Mathematics*, Vol. 2 (1901), p. 78.

†Since the numbers are squares whenever the corresponding substitutions are squares.

‡This result can also be seen directly from the substitutions of the given cyclic group of order $p^{\alpha-1}(p-1)$.

§Zolotareff, *Nouvelles Annales*, Vol. 11 (1872), p. 354.

We may again associate with each substitution of I the index (modulo $p^\alpha q^\beta$) of the power into which the substitution transforms all the substitutions of C. If this is done Legendre's law of quadratic reciprocity says that the substitution which corresponds to $k \equiv p+q$ modulo $p^\alpha q^\beta$ is positive unless p and q are both of the form $4n+3$. It is clear that the positive substitutions of I correspond to numbers which are either quadratic residues of both p and q, or quadratic non-residues of both p and q. The negative substitutions of I correspond to numbers which are quadratic residues of only one of the two numbers p, q.

In abstract group theory the following direct application of the law of quadratic reciprocity is evident from the preceding developments. If the group of isomorphisms I of the cyclic group of order p^α is represented as a number group modulo p^α all the numbers which are in the subgroup of order $\frac{p^{\alpha-1}(p-1)}{2}$ are quadratic residues of p while the remaining numbers are non-residues. All the numbers of I may be represented by primes in an infinite number of ways since they are only determined with respect to modulus p^α . Legendre's law of quadratic reciprocity states that if q is any prime in the subgroup of I whose order is $\frac{p^{\alpha-1}(p-1)}{2}$ then will the numbers which are congruent to p mod q be in the subgroup of order $\frac{q^{\beta-1}(q-1)}{2}$ of the group of isomorphisms of the cyclic group of order q^β , unless both p and q are of the form $4n+3$.

Since the subgroup of order p^{α_1}, $\alpha_1 < \alpha$, contained in I is composed of the numbers which are congruent to 1 modulo $p^{\alpha-\alpha_1}$, it follows that the subgroup of order $\frac{p^{\alpha_1}(p-1)}{d}$ contained in I is composed of the numbers which are congruent modulo $p^{\alpha-\alpha_1}$ to the numbers of the subgroup of order $\frac{p-1}{d}$. When $d=2$ the latter numbers are the quadratic residues of p modulo p. If we multiply these numbers by the numbers in the subgroup of order p^{α_1} the products are still quadratic residues of p. By making $\alpha_1 = \alpha - 1$ we have another proof of the theorem that a quadratic residue of p if also a quadratic residue of p^α . Another particular case is that *every subgroup of I whose order is divisible by $p^{\alpha-1}$ includes the numbers which are congruent to any of its numbers modulo p.*

While we know some of the properties of the numbers in the subgroups of I, little is known in reference to what particular numbers occur in these subgroups. It is evident that $p^\alpha - 1$ occurs in every subgroup of even order. That is, when the group constituted by the quadratic residues of p^α is of even order, -1 is a quadratic residue of p and vice versa. This follows directly from the fact that -1 corresponds to the operator of order 2 in I. The numbers 1 and -1 are the only ones whose orders are independent of the value of the modulus.

As has been observed above the numbers whose orders are given powers of p can also be directly written down. The determination of the numbers whose orders divide $p-1$ presents difficulties which have not yet been surmounted. If

we knew the latter numbers and their orders, we would know the orders of all the numbers, since the order of the product of two such numbers is the product of their orders. In the case when $p=3$, the operators whose orders divide $p-1$ are 1 and p^a-1. Hence, in this case, the order of every number is known. In particular, the primitive roots of p^a are the products of p^a-1 into the numbers of the form $1+kp$ (k being prime to p), as is well known.

Even the number 2 presents difficulties which have not been overcome. That is, we do not know the order of the operator which transforms each operator of C into its square. The well known number theory results along this line; that is, those which relate to the quadratic character of 2, may be stated in group theory language as follows: In the number group formed by the natural numbers which are prime to p and less than p^a, the order of 2 divides $\frac{p^{a-1}(p-1)}{2}$ only when p is of the form $8n\pm1$. When p is not of this form the order of 2 must involve the highest power of 2 contained in $p-1$.

A SHORT PROOF FOR THE NUMBER OF TERMS IN A DETERMINANT WHICH ARE INDEPENDENT OF THE ELEMENTS OF THE PRINCIPAL DIAGONAL.

By ORLANDO S. STETSON, Syracuse University.

The problem of finding the number of terms, $\varphi(n)$, in the given determinant which are independent of the elements of the principal diagonal may be reduced to the question of finding the number of terms in the expansion of the *invertebrate* determinant Δ_n (second formula for $k=n$, MONTHLY, 1904, page 167). Hence

$$\varphi(n)=n!-n(n-1)!+\frac{n(n-1)}{1.2}(n-2)!-\frac{n(n-1)(n-2)}{1.2.3}(n-3)!$$

$$+\ldots+(-1)^{n-1}\frac{n(n-1)(n-2)\ldots3.2}{1.2.3\ldots(n-1)}(1)!+(-1)^n\frac{n(n-1)(n-2)\ldots3.2.1}{1.2.3\ldots n}$$

Removing from each of the terms the factor $n!$ and noticing that the first two terms are equal but opposite in sign, we have

$$\varphi(n)=n!\left[\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}-\ldots+(-1)^n\frac{1}{1.2.3\ldots n}\right].$$

NOTE ON THE nth DERIVATIVE OF A DETERMINANT WHOSE CONSTITUENTS ARE FUNCTIONS OF A GIVEN VARIABLE.*

By W. J. RUSK. Grinnell, Iowa.

Let the determinant be $D=(a_1 b_2 \ldots\ldots l_r)$ where the $a, b, \ldots\ldots, l$ are functions of a variable t. Then

$$\frac{dD}{dt}=(a_1' \, b_2 \ldots\ldots l_r)+\ldots\ldots+(a_1 \, b_2 \ldots\ldots l_r').$$

$$\frac{d^2 D}{dt^2}=(a_1'' \, b_2 \ldots\ldots l_r)+\ldots\ldots+(a_1 \, b_2 \ldots\ldots l_r'')$$

$$+2(a_1' \, b_2' \, c_3 \ldots\ldots l_r)+\ldots\ldots+2(a_1 \, b_2 \ldots\ldots k'_{r-1} \, l_r').$$

Consider now the expansion

$$(a+b+c+\ldots\ldots+l)^2=a^2 \, b^0 \, c^0 \ldots\ldots l^0+a^0 \, b^2 \, c^0 \ldots\ldots l^0+2a^1 \, b^1 \, c^0 \ldots\ldots l^0+\ldots\ldots$$

If we interpret the power of a^n as the nth derivative of a and write instead of $a^2 \, b^0 \, c^0 \ldots\ldots l^0$ the expression $(a_1'' \, b_2 \, c_3 \ldots\ldots l_r)$, etc., we have, symbolically,

$$\frac{d^2 D}{dt^2}=(a+b+c+\ldots\ldots+l)^2.$$

Suppose now that, symbolically,

$$\frac{d^{n-1} D}{dt^{n-1}}=(a+b+c+\ldots\ldots+l)^{n-1}=\sum \frac{(n-1)!}{\alpha! \, \beta! \ldots\ldots \lambda!} a^\alpha \, b^\beta \ldots\ldots l^\lambda ,$$

where $\alpha+\beta+\gamma+\ldots\ldots+\lambda=n-1$.

Now suppose the symbolic form be interpreted as before and another differentiation with respect to t carried out. Then the coefficient of $a^{\alpha'} \, b^{\beta'} \ldots\ldots l^{\lambda'}$ will be

$$C=\frac{n!}{\alpha'! \, \beta'! \ldots\ldots \lambda'!},$$

where $\alpha'+\beta'+\ldots\ldots+\lambda'=n$. For this term can be obtained by differentiation from terms with exponents one less than $\alpha', \beta', \ldots\ldots$, or λ', and its coefficient will be

$$\frac{(n-1)!}{(\alpha'-1)! \, \beta'! \ldots\ldots \lambda'!}+\frac{(n-1)!}{\alpha'! \, (\beta'-1)! \ldots\ldots \lambda'!}+\ldots\ldots=C.$$

*Presented to the American Mathematical Society (Chicago), April, 1904.

ON THE CYCLOTOMIC FUNCTION.*

By DR. L. E. DICKSON, The University of Chicago.

1. If $p^n=1$ and $p^m \neq 1$ $(m<n)$, p is called a primitive nth root of unity. Let $Q_n(x)$ be the equation whose roots are the various nth primitive roots of unity without repetition. Let $n=\nu p^r$, ν not being divisible by the prime p. We first prove that

$$(1) \qquad Q_n(x)=Q_\nu\left(x^{p^r}\right) \div Q_\nu\left(x^{p^{r-1}}\right).$$

To show that the division is exact, let $\xi_1, \ldots, \xi_e$ be the distinct νth roots of unity. Then $\xi_1{}^p, \ldots, \xi_e{}^p$ differ only as to order from $\xi_1, \ldots, \xi_e$. Hence

$$Q_\nu\left(x^{p^r}\right)=\prod_{i=1}^{e}\left(x^{p^r}-\xi_i{}^p\right), \quad Q_\nu\left(x^{p^{r-1}}\right)=\prod_{i=1}^{e}\left(x^{p^{r-1}}-\xi_i\right).$$

Since $y-\xi$ divides $y^p-\xi^p$, the division (1) is exact. If the value x makes the quotient vanish, then $(x^{p^r})^m=\xi_i{}^m=1$ if and only if m is a multiple of ν, while $x^{\nu p^{r-1}} \neq 1$; hence x is a primitive nth root of unity.

We employ (1) as a recursion formula to determine $Q_n(x)$. As a permanent notation, set $n=p_1{}^{r_1} p_2{}^{r_2} \ldots p_s{}^{r_s}$, where $p_1, \ldots, p_s$ are distinct primes. Then

$$(2) \quad Q_n(x)=Q_\mu\left(x^{p_1^{r_1} p_2^{r_2}}\right) Q_\mu\left(x^{p_1^{r_1-1} p_2^{r_2-1}}\right) \div Q_\mu\left(x^{p_1^{r_1} p_2^{r_2-1}}\right) Q_\mu\left(x^{p_1^{r_1-1} p_2^{r_2}}\right),$$

where $\mu=n \div p_1{}^{r_1} p_2{}^{r_2}$. In general, for $N=p_1{}^{r_1} p_2{}^{r_2} \ldots p_\sigma{}^{r_\sigma}$, we get

$$(3) \qquad Q_n(x)=\frac{Q_{n/N}(x^N) \prod Q_{n/N}(x^{N/p_i p_j}) \ldots}{\prod Q_{n/N}(x^{N/p_i}) \prod Q_{n/N}(x^{N/p_i p_j p_k}) \ldots},$$

in which $i, j, \ldots$ range from $1, \ldots, \sigma$. Now $Q_1(x)=x-1$. For $\sigma=s$, $N=n$, and (3) becomes

$$(4) \qquad Q_n(x)=\frac{(x^n-1) \prod (x^{n/p_i p_j}-1) \ldots}{\prod (x^{n/p_i}-1) \prod (x^{n/p_i p_j p_k}-1) \ldots},$$

where in the denominator the products extend over the combinations 1, 3, 5, ... at a time of $p_1, \ldots, p_s$; in the numerator, 2, 4, ... at a time.

Conversely, (1) follows from (4). The terms of (4) in which p_1 does not enter explicitly combine into $Q_\nu\left(x^{p_1^{r_1}}\right)$; those in which p_1 enters explicitly combine into $1 \div Q_\nu\left(x^{p_1^{r_1-1}}\right)$.

*Read before the American Mathematical Society, Chicago, April 22, 1905.

2. The usual proof* of (4) is essentially only a verification. Since $Q_d(x)=0$ gives all the primitive dth roots of unity withour repetition, we have

$$(5) \qquad x^n-1=\Pi Q_d(x), \quad x^{n/p_1}-1=\Pi Q_\delta(x), \ldots$$

where d ranges over all the divisors of n; δ over those of n/p_1. When the products (5) are substituted in the second member of (4), every Q cancels except Q_n. In fact, if $d=p_1^{a_1}\ldots p_t^{a_t}p_{t+1}^{r_{t+1}}\ldots p_s^{r_s}$, where $t>0$ and each $a_i<r_i$, Q_d divides exactly $A=1+{}_tC_2+{}_tC_4+\ldots$ terms of the numerator of (4) and exactly $B={}_tC_1+{}_tC_3+\ldots$ terms of the denominator, ${}_tC_k$ being the number of combinations of t things k at a time. But $A-B=(1-1)^t=0$.

3. From equation (4) follows as a corollary the important formula

$$(6) \qquad \phi(n)=n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_s}),$$

where $\phi(n)$ denotes the number of positive integers not greater than n and relatively prime to n. Indeed, if ρ is a primitive nth root of unity, ρ^m is likewise if and only if m is relatively prime to n. But the degree of (4) evidently equals the right member of (6).

It follows from (4) that the polynomial $Q_n(x)$ has integral coefficients.

There are various proofs of the theorem that $Q_n(x)$ is algebraically irreducible, *i. e.*, can not be expressed as a product of polynomials in x with rational coefficients.†

4. Theorem. *For an integer x, the greatest common divisor g of $Q_n(x)$ and $x^{n/p_1}-1$ is 1 or p_1. If $g=p_1$, Q_n is not divisible by p_1^2 unless $n=p_1=2$, $x\equiv 3\ (mod 4)$, whence $Q_n=x+1$.*

Dividing the first equation (5) by the second, we get

$$(7) \qquad (x^{n/p_1})^{p_1-1}+(x^{n/p_1})^{p_1-2}+\ldots+x^{n/p_1}+1=Q_n(x).P(x),$$

$P(x)$ being a polynomial in x with integral coefficients. When the left member of (7) is divided by $x^{n/p_1}-1$, the remainder is 1 or p_1. Hence $g=1$ or p_1.

Let $g=p_1$, so that $x^{n/p_1}-1=kp_1$, k an integer. Substituting kp_1+1 for x^{n/p_1} in (7), we obtain $p_1+\frac{1}{2}p_1(p_1-1)kp_1+$terms in p_1^2. This is not divisible by p_1^2 if $p_1>2$, nor if $p_1=2$ and k is even. If $p_1=2$ and $k=2l+1$, then $x^{n/2}=4l+3$, whence $n/2$ must be odd and $x\equiv 3$ (mod4). Suppose that $n>2$ and $n/2=p^r p_3^{r_3}\ldots p_s^{r_s}=m=$odd. Performing in (4) the divisions of the type $(x^{2a}-1)\div(x^a-1)$, we get

$$(8) \qquad \frac{x^m+1}{x^{m/p}+1}\cdot\prod_{i=3}^{s}\frac{x^{m/p\,p_i}+1}{x^{m/p_i}+1}\cdot\prod_{i,j=3}^{s}\frac{x^{m/p_ip_j}+1}{x^{m/p\,p_ip_j}+1}\cdot\prod_{i,j,k}\frac{x^{m/p\,p_ip_jp_k}+1}{x^{m/p_ip_jp_k}+1}\ldots$$

*On the general principle of the inversion involved, see Dedekind, *Crelle*, Vol. 54 (1857), pp. 1-26; Dirichlet-Dedekind, *Zahlentheorie*, p. 362; Bachmann, *Kreistheilung*, 1872, pp. 8-11, 16, and *Zahlentheorie*, I, pp. 40-42.

†See Bachmann, *Kreistheilung* (Leipzig. Teubner, 1872), pp. 31-43.

Since the exponents are all odd, each fraction or its inverse equals $1+f$, f containing an even number of powers of x. Hence Q is odd (cf. §5).

5. Theorem. *For $n=p_1^{r_1}\ldots p_s^{r_s}$ and x an integer, $Q_n(x)$ is divisible by p_1 if and only if x belongs to the exponent $\nu=n/p_1^{r_1}$ modulo p_1; in the contrary case, $Q_n(x)\equiv 1\ (mod\ p_1)$.*

By Fermat's theorem, $x^{p_1}\equiv x \pmod{p_1}$. Hence by (1), $Q_n(x)\equiv 1 \pmod{p_1}$ unless $Q_\nu(x)\equiv 0$. Now $Q_\nu(x)$ divides algebraically the function

$$(x^\nu-1)\div(x^{\nu/p_i}-1)=(x^{\nu/p_i})^{p_i-1}+\ldots+x^{\nu/p_i}+1 \qquad (1<i\leqq s).$$

Hence if $x^{\nu/p_i}\equiv 1 \pmod{p_1}$, there is an integer k such that $kQ_\nu(x)\equiv p_i \pmod{p_1}$; whence Q_ν is not congruent to 0 (mod p_1). There remains the case in which x^{ν/p_i} is not congruent to 1 (mod p_1) for $i=2, \ldots, s$. If $x^\nu\equiv 1 \pmod{p_1}$, x belongs to the exponent ν modulo p_1 and $Q_\nu\equiv 0$; if $x^\nu-1$ is not congruent to 0, its divisor Q_ν is not congruent to 0 (mod p_1).

Example. For $n=2.3.7$, formula (8) gives

$$Q_{42}(x)=\frac{(x^{21}+1)\ (x+1)}{(x^7+1)\ (x^3+1)}=x^{12}+x^{11}-x^9-x^8+x^6-x^4-x^3+x+1.$$

Thus $Q_{42}\equiv 1$ (mod 2 or 3); $Q_{42}(x)\equiv 1$ (mod 7) if $x\equiv 0, -1$ or $x^3\equiv +1$; but $Q_{42}(x)\equiv 0$ (mod 7) if $x^2-x+1\equiv 0$ (mod 7), *i. e.*, if x belongs to the exponent $\nu=6$.

Corollary. No one of the prime factors of n except the greatest can divide $Q_n(x)$.

6. Theorem. *If x is a positive integer >1, $Q_n(x)$ has a prime factor not dividing x^m-1 $(m<n)$, except in the cases $n=2$, $x=2^k-1$ $(k\geqq 2)$; and $n=6$, $x=2$.*

If $n=p^r$, $Q=y^{p-1}+\ldots+y+1>p$, where $y=x^{p^{r-1}}$, so that the theorem follows from §4. We suppose henceforth that $n=p_1^{r_1}\ldots p_s^{r_s}$, $s\geqq 2$.

In view of §1, $Q_n=A/B$, where

$$A=\frac{x^n-1}{x^{n/p_1}-1}\cdot\Pi\frac{x^{n/p_ip_j}-1}{x^{n/p_1p_ip_j}-1}\ldots,\quad B=\Pi\frac{x^{n/p_i}-1}{x^{n/p_1p_i}-1}\cdot\Pi\frac{x^{n/p_ip_jp_k}-1}{x^{n/p_1p_ip_jp_k}-1}\ldots,$$

in which $i, j, k, \ldots$ run from 2 to s. Now $x^{a(k-1)}<(x^{ka}-1)\div(x^a-1)<2x^{a(k-1)}$. Hence $Q_n>\alpha/\beta$, where α is the result of retaining only the first term of each division in A, β the result of taking twice the first term of each division in B. The number of factors 2 introduced in B is ${}_{s-1}C_1+{}_{s-1}C_3+\ldots=2^{s-2}$. The exponent of x in α/β is evidently the degree $\phi(n)$ of Q_n. Let $x^{n/p_1p_2\cdots p_s}=y$. Hence

$$Q_n(x)>y^{(p_1-1)\ldots(p_s-1)}\div 2^{2^{s-2}},$$

y an integer>1. In view of § §4-5, it suffices to prove that $Q_n>p_1$, the greatest of the primes p_i, the case $n=6$, $x=2$ being an exception. For $s>2$, we have $p_1\geqq 5$, $y^{p_1-1}>2p_1$, the latter being true for $y=2$. Hence

$$Q_n > (2p_1)^{(p_2-1)\cdots(p_s-1)} \div 2^{2^{s-2}} > p_1,$$

since at least $s-2$ of the primes $p_2, \ldots, p_s$ exceed 2, so that the exponent is $\overline{\gtrless} 2^{s-2}$. For $s=2$, we have $p_1 \overline{\gtrless} 3$, $Q_n > \frac{1}{2} y^{(p_1-1)(p_2-1)} \overline{\gtrless} p_1$ unless $p_1=3$, $y^{p_2-1}=2$, whence $p_2=2$, $y=2$, $n=6$, $x=2$.

Corollary. *If x is a positive integer >1, x^n-1 has a prime factor not dividing x^m-1 $(m<n)$, except in the cases $n=2$, $x=2^k-1$; $n=6$, $x=2$.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problems 219, 220 were also solved by L. E. Newcomb. No. 222 was also solved by A. H. Holmes.

223. Proposed by THEODORE L. DE LAND, Office of the Secretary of the Treasury, Washington, D.C.

An officer in the Treasury Department assigned three clerks to count a lot of silver dollars and when finished noted that there was an apparent difference in their efficiency; and, to determine the fact, gave to each a similar lot of the same amount to count, the only record made at the time being that A to count his lot alone, took three weeks longer, B took two weeks longer, and C took one week longer than it took for all working together to count the first lot. The best counter, on the record made, was given an efficiency mark of 93 on the scale of 100. What efficiency mark should, on the record, be given to each of the other two counters?

Solution by the PROPOSER.

Let x=the time for A, B, and C working together to finish one lot.
Then $x+3$=the time for A to finish one lot working alone;
$x+2$=the time for B to finish one lot working alone; and
$x+1$=the time for C to finish one lot working alone.

Then $\frac{1}{x}$=what A, B, and C can do in one week working together;

$\frac{1}{x+3}$=what A can do in one week working alone;

$\frac{1}{x+2}$=what B can do in one week working alone; and

$\frac{1}{x+1}$=what C can do in one week working alone.

Equating like terms we have,

$$\frac{1}{x}=\frac{1}{x+3}+\frac{1}{x+2}+\frac{1}{x+1} \quad \text{.........(1).}$$

Reducing, we have,

$$x^3+3x^2-3=0 \ldots\ldots (2).$$

By Horner's method, we have from equation (2), $x=0.879385+$. Therefore

$$\frac{1}{x+3}=\frac{1}{3.839385};\quad \frac{1}{x+2}=\frac{1}{2.879385};\text{ and }\frac{1}{x+1}=\frac{1}{1.879385}.$$

It is evident that C is the best clerk and was given the 93% on the efficiency record. The records should be inversely proportional to the time expended for equivalent work. In order to compare C and B, and C and A, we have

$$x+2 : x+1=93\% : B\text{'s mark};$$
$$x+3 : x+1=93\% : A\text{'s mark};$$

and therefore,

$$2.879385 : 1.879385=93\% : 60.70\%=B\text{'s mark; and}$$
$$3.879385 : 1.879385=93\% : 45.05\%=A\text{'s mark.}$$

Thus, if C were given on the efficiency record 93%, A should be given 45.05%, and B should be given 60.70%.

Also solved by G. B. M. Zerr, S. A. Corey, G. W. Greenwood, F. D. Whitlock, R. D. Carmichael, A. H. Holmes, and J. Scheffer.

224. Proposed by G. W. GREENWOOD, M. A. (Oxon). Lebanon, Ill.

Show that, if none of the quantities x, y, z is zero, the result of eliminating them from

$$(x+y)(x+z)=bcyz \ldots\ldots (1),$$
$$(y+z)(y+x)=cazx \ldots\ldots (2),$$
$$(z+x)(z+y)=abxy \ldots\ldots (3),$$

is $\begin{vmatrix} \pm a, & 1, & 1 \\ 1, & \pm b, & 1 \\ 1, & 1, & \pm c \end{vmatrix}=0.$ [Oxford, 1896.]

Solution by C. H. MILLER, West Point. N. Y., and the PROPOSER.

By multiplying the second equation by the third, dividing by the first, and transposing, we obtain

$$\pm ax+g+z=0.$$

From this, and two similar equations, we get the required elimininant.

Also solved by J. B. Faught, G. B. M. Zerr, R. D. Carmichael, J. Scheffer, and J. O. Mahoney.

225. Proposed by H. M. ARMSTRONG, Cooch's Bridge, Delaware.

If $\alpha=ax+cy+bz \ldots\ldots (1)$, $\beta=cx+by+az \ldots\ldots (2)$, $\gamma=bx+ay+cz \ldots\ldots (3)$, show that $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(a^3+b^2+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

Solution by the PROPOSER.

The required result follows directly from the equality,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \cdot \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Also solved by J. B. Faught, G. B. M. Zerr, G. W. Greenwood, Grace M. Bareis, J. O. Mahoney, F. D. Posey, F. O. Whitlock, J. Scheffer.

*** Dr. L. E. Dickson points out that a similar theorem holds for any determinant whose matrix is the body of a multiplication-table of a finite group.

GEOMETRY.

251. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Represent the vertices of any regular polygon by the consecutive numbers 1, 2.....p.....q.....r.....n. To find the sides and area of the triangle formed by joining p, q, and r.

Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill., and A. H. HOLMES, Brunswick, Me.

The central angles subtended by the chords (pq) and (qr) are respectively,

$$2(q-p)\frac{\pi}{n} \text{ and } 2(r-q)\frac{\pi}{n}.$$

The angle pqr is found to be $\pi-(r-p)\dfrac{\pi}{n}$. Hence the required area is

$$\tfrac{1}{2}.pq.qr.\sin\angle pqr=2a^2\sin(q-p)\frac{\pi}{n}.\sin(r-q)\frac{\pi}{n}.\sin(r-p)\frac{\pi}{n},$$

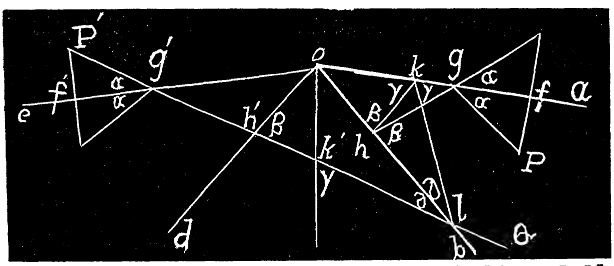
where a is the radius of the circum-circle of the polygon.

252. Proposed by FREDERICK R. HONEY, Ph. B., Trinity College, Hartford, Conn.

Two plane mirrors form an angle which is less than 45°. Any two points are assumed within this angle in a plane perpendicular to the intersection of the mirrors. A ray of light passes through one point, and after being reflected twice at each mirror, it passes through the second point. Find the path of the ray.

Solution by R. A. WELLS, Westminster College, Fulton, Mo.; THEODORE LINQUIST, Wahpeton, N. D.; and the PROPOSER.

Let oa and ob represent the mirrors; and P and Q the assumed points. Draw oc, od, and oe, making each of the angles boc, cod, and doe equal to aob. Draw Pf perpendicular to oa. Make $of'=of$; and draw $f'P'$ perpendicular to oe and equal to Pf. Draw QP', intersecting ob at l, oc at k', od at h', and oe at g'. Make $og=og'$; $oh=oh'$; $ok=ok'$. Join Pg, gh, hk, and kl. $PghklQ$ is the path of the ray.



The Greek letters indicate the equality of certain angles, and will assist the reader in the demonstration.

Also solved by G. W. Greenwood.

The following contributors sent in solutions to this department too late for credit in the last issue: G. B. M. Zerr solved 245; Theodore Linquist, 248 and 249; A. H. Holmes, 248, 249, and 250.

253. Proposed by SAM I. JONES, Gunter Bible College, Gunter, Texas.

The number of cubic inches contained by two equal opposite spherical segments, together with the number of cubic inches contained by the cylinder included between these segments, is 600; if this be $\frac{5}{8}$ of the number of cubic inches contained by the whole sphere, find the height of the cylinder.

Solution by THEODORE LINQUIST, Wahpeton, N. Dak.; G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill., and A. H. HOLMES, Brunswick, Me.

Let R=the radius of the sphere, and $2h$ the altitude of the cylander. Then $R-h$=the altitude of the segment of the sphere, and $\sqrt{(R^2-h^2)}$ is the radius of the base of the segment and the radius of the cylinder.

The volume of the two segments$=2[\frac{1}{6}\pi(R-h)^3+\frac{1}{2}\pi(R-h)(R^2-h^2)]$, and the volume of cylinder$=2\pi h(R^2-h^2)$.

$\therefore \frac{4}{3}\pi(R^3-h^3)$=the volume of the segments, and the cylinder$=\frac{5}{8}(\frac{4}{3}\pi R^3)$, by the conditions of the problem.

$\therefore 3h^3=R^3$. $\therefore \frac{4}{3}\pi(R^3-h^3)=\frac{8}{3}\pi h^3=600$, by the conditions of the problem.

$\therefore 2h=2\sqrt[3]{(225/\pi)}$.

Also solved by J. Scheffer.

CALCULUS.

191. Proposed by J. E. SANDERS, Hackney, Ohio.

A fly goes along a radius of a moving carriage wheel from center to circumference while the wheel makes n revolutions. If each move uniformly, what is the equation to the curve described by the fly in space, and what is its length when the wheel has made $1/m$ of a revolution?

Solution by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.

Take the path of the center of the wheel as x-axis, and the initial point as origin. Let the fly move on a radius making, initially, an angle ϕ with this axis. Denote the radius by a. Let C be the position of the center of the wheel, and P be that of the fly after the wheel has turned through an angle ω. Then

$$OC=a\omega,\ CP=\frac{a\omega}{2n\pi},$$

and the coördinates of the position of P are

$$x=a\omega\left(1+\frac{\cos(\omega+\phi)}{2n\pi}\right),\ y=\frac{a\omega\sin(\omega+\phi)}{2n\pi}.$$

*** An excellent solution was received from Professor Zerr. He takes the horizontal line on which the wheel travels as the x-axis, and gets for the equation of the path of the fly,

$$x=a\theta\left(1-\frac{\sin\theta}{2n\pi}\right),\quad y=a\left(1-\frac{\theta\cos\theta}{2n\pi}\right),\text{ for required length.}$$

$$s=a\int_0^{2\pi/m}\sqrt{1+\frac{1+\theta^2}{4\pi^2n^2}-\frac{\sin\theta+\theta\cos\theta}{\pi n}}\,d\theta. \qquad \text{F.}$$

192. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Show that the volume V of the hyper-ellipsoid with semi-axes a_1, a_2, a_3, a_4, etc., in space of $2n$ and $2n+1$ dimensions is

$$V_{2n}=\frac{a_1.a_2.a_3.....a_{2n}.\pi^n}{1.2.3.4.5..............n};\qquad V_{2n+1}=\frac{2^{n+1}.a_1.a_2.a_3.........a_{2n+1}.\pi^n}{1.3.5.7.9.................(2n+1)}.$$

Solution by the PROPOSER.

Let $\left(\frac{x_1}{a_1}\right)^2+\left(\frac{x_2}{a_2}\right)^2+\left(\frac{x_3}{a_3}\right)^2+.....+\left(\frac{x_r}{a_r}\right)^2=1$ be the equation to the hyper-ellipsoid. Then its volume is $V=2^r\int\int\int.....dx_1\,dx_2\,dx_3.....dx_r$.

Let $x_1/a_1=y_1$, $x_2/a_2=y_2$, ..., $x_r/a_r=y_r$.

$\therefore V=2^ra_1a_2a_3.....a_r\int\int\int.....dy_1\,dy_2\,dy_3.....dy_r$, subject to the condition, $y_1^2+y_2^2+y_3^2+.....+y_r^2=1$.

$\therefore V=\dfrac{a_1a_2a_3.....a_n[\Gamma(\frac{1}{2})]^r}{\Gamma(1+\frac{1}{2}r)}$.

When $r=2n$,

$$V=\frac{a_1a_2a_3.....a_{2n}\,\pi^n}{1.2.3.4.....n}.$$

When $r=2n+1$,

$$V=\frac{a_1a_2a_3.....a_{2n+1}\,\pi^n\,\Gamma(\frac{1}{2})}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.....\frac{2n+1}{2}\Gamma(\frac{1}{2})}=\frac{2^{n+1}a_1a_2a_3.....a_{2n+1}\,\pi^n}{1.3.5.7.9.....(2n+1)}.$$

193. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the eccentricity of the maximum semi-ellipse inscribed in a given isosceles triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. SCHEFFER, Hagerstown, Md.

Let the mid-point of the base be the origin, a=altitude, b=base of triangle. Let $x^2/m^2+y^2/n^2=1$ be the ellipse. Then πmn=maximum.

$\therefore n/m=y/x$.

Let (h, k) be the tangent point of the ellipse with a side.

Then $a=n^2/k$, $\frac{1}{2}b=m^2/n$, or $k/h=2a/b=n/m$. Also $2ah+bk=hk$.

$\therefore k=\frac{1}{2}a$, $h=\frac{1}{4}b$. $\therefore b^2n^2+4a^2m^2=16m^2n^2$. And $bd=2am$.

$\therefore m=\dfrac{b}{2\sqrt{2}}$, $n=\dfrac{a}{\sqrt{2}}$; $(\text{eccentricity})^2=\dfrac{4a^2-b^2}{4a^2}$.

II. Solution by A. H. HOLMES, Brunswick, Me.

Let $2a$=base of the isosceles triangle, and b its perpendicular height.

Construct on $2a$ an equilateral triangle, and inscribe in it a semi-circle its diameter collinear with base $2a$. Then the radius of the semi-circle will be $\dfrac{a\sqrt{3}}{2}$ which is one-half the perpendicular of the equilateral triangle. Now consider this triangle to be projected into an isosceles triangle whose base will be, of course, the same as that of the equilateral triangle, but whose perpendicular height is b. The semi-circle inscribed in the equilateral triangle will be projected into the maximum semi-ellipse that can be inscribed in the isosceles triangle, and one of its semi-axes will have the same proportion to the perpendicular of the isosceles triangle that the radius of the semi-circle has to the perpendicular of the equilateral triangle.

$\therefore$ Eccentricity of ellipse$=\dfrac{\sqrt{(b^2-3a^2)}}{b}$ or $\dfrac{\sqrt{(3a^2-b^2)}}{a\sqrt{3}}$, accordingly as $\sqrt{(a^2+b^2)}$ is greater or less than $2a$. If b=one of the sides,

$$\varepsilon=\sqrt{\frac{b^2-4a}{b^2-a^2}},\quad\text{or}\quad\frac{\sqrt{(4a^2-b^2)}}{a\sqrt{3}}.$$

Also sǒlved by Jacob Westlund.

DIOPHANTINE ANALYSIS.

123. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Of two numbers $a_ib_ic_id_ie_i$ $(i=1, 2)$ it is given that their 10 digits $a_1, \ldots, e_2$ form a permutation of 0, 1, ..., 9, and that the sum of the two is $x3951$. Give an immediate evaluation of x; also list the possible pairs a_1, a_2; .....; e_1, e_2.

Solution by the PROPOSER.

Since the sum of the 10 digits is 45, $x+18$ must be a multiple of 9 by the rule of casting out of 9's. Hence $x=9$.

Next, on adding the third column there cannot be 1 to carry; otherwise c_1+c_2 or c_1+c_2+1 would be 19, and $c_1 \overline{\gtrless} 9$, $c_2 \overline{\gtrless} 9$. Hence

$$(1)\quad b_1+b_2=3,\ a_1+a_2=9;\ \text{or}\ (2)\quad b_1+b_2=13,\ a_1+a_2=8.$$

If $e_1+e_2=1$, the b's are not 0, 3; nor 1, 2. Hence in this case,

$e_1, e_2=0, 1$; $b_1+b_2=13$; $a_1+a_2=8$; $d_1+d_2=5$, $c_1+c_2=9$; or $d_1+d_2=15$, $c_1+c_2=8$.

Thus a_1, a_2=2, 6 or 3, 5; $d_1+d_2=15$, giving sets I, II below. Let next $e_1+e_2=11$. Then $d_1+d_2=4$, $c_1+c_2=9$; or $d_1+d_2=14$, $c_1+c_2=8$. From these and (1) or (2), we get III.....XI as the only sets.

	a_1, a_2	b_1, b_2	c_1, c_2	d_1, d_2	e_1, e_2
I	2, 6	4, 9	3, 5	7, 8	0, 1
II	3, 5	4, 9	2, 6	7, 8	0, 1
III	1, 8	0, 3	2, 6	5, 9	4, 7
IV	3, 6	1, 2	0, 8	5, 9	4, 7
V	4, 5	0, 3	1, 7	6, 8	2, 9
VI	0, 9	1, 2	3, 5	6, 8	4, 7
VII	1, 7	5, 8	3, 6	0, 4	2, 9
VIII	2, 6	5, 8	0, 9	1, 3	4, 7
IX	3, 5	6, 7	1, 8	0, 4	2, 9
X	0, 8	4, 9	2, 7	1, 3	5, 6
XI	0, 8	6, 7	4, 5	1, 3	2, 9

Sets VI, X, and XI may properly be excluded.

Also solved by G. B. M. Zerr.

MISCELLANEOUS.

132. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Six officers of different grades (1, 2, 3, 4, 5, 6) from each of six branches the army (a, b, c, d, e, f) are to be arranged in a square so that each rank and each file shall have an officer of each grade and each branch. Can it be done? If not, prove it. The arrangement of five officers of each kind is easy.

Remark by L. E. DICKSON, The University of Chicago.

This problem, proposed in the February, 1903, number, is here repeated to call attention to the fact that no solution has yet been sent to the editors. If, instead of 6, we employ an odd number n, we obtain an immediate solution with $a_1b_2c_3.....\nu_n$ as the first row, $a_1a_na_{n-1}.....a_3a_2$ as the main diagonal, the scheme being completed by permuting a, b, c, ..., ν cyclically, and 1, 2, ..., n cyclically. Thus for $n=3$ we obtain the (single, notation apart) possible solution:

$$\begin{matrix} a_1 & b_2 & c_3 \\ c_2 & a_3 & b_1 \\ b_3 & c_1 & a_2. \end{matrix}$$

The problem is impossible for $n=2$. I proceed to show that there are exactly two distinct solutions for $n=4$. I first find the possible schemes for the *letters*.

By interchange of columns, we may bring the a's into the main diagonal. Call the first row *abcd*. If b is fourth in the second row, we interchange the third and fourth row, the third and fourth columns, and permute c, d, and get *abcd* as

the new first row, a's in diagonal, and b third in second row. The scheme is then necessarily (I). If b is first in the second row, the scheme is either (II) or (III).

$$\text{(I)}\ \begin{matrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{matrix} \qquad \text{(II)}\ \begin{matrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{matrix} \qquad \text{(III)}\ \begin{matrix} a & b & c & d \\ b & a & d & c \\ d & c & a & b \\ c & d & b & a \end{matrix}$$

If in (III) we interchange the second and third columns, and also rows, and permute b, c, we get (I).

We attach the subscripts to the letters of the first row in the order 1, 2, 3, 4. The diagonal terms must be $a_1a_4a_2a_3$ or $a_1a_3a_4a_2$. For (I), b in the second row must be b_1; in the former case, d in the fourth row must be d_4 contrary to d of the first row; in the latter case, d in the fourth row must be d_2, contrary to a of the fourth row. Hence (I) is excluded. For (II) the two schemes are evidently

$$(A)\ \begin{matrix} a_1 & b_2 & c_3 & d_4 \\ b_3 & a_4 & d_1 & c_2 \\ c_4 & d_3 & a_2 & b_1 \\ d_2 & c_1 & b_4 & a_3 \end{matrix} \qquad (B)\ \begin{matrix} a_1 & b_2 & c_3 & d_4 \\ b_4 & a_3 & d_2 & c_1 \\ c_2 & d_1 & a_4 & b_3 \\ d_3 & c_4 & b_1 & a_2 \end{matrix}$$

If we view the square (A) from the side, instead of the top, we get (B). If we reflect (A) on the main diagonal and then permute 2, 4, 3 cyclically, we obtain (B). But by no change of notation of letters or subscripts is (A) converted into (B). Note that the arrangements of the letters (as well as the subscripts) in (A) or (B) define the non-cyclic group of order 4.

These results for $n=4$ and n odd suggest that for $n=6$ the arrangements of the letters and subscripts might be derivable from those in the first row by means of the substitutions of the same regular group on six letters. But this is readily verified to be impossible. Hence if there is a solution for $n=6$, it is not a group solution of the type mentioned.

PROBLEMS FOR SOLUTION.

ALGEBRA.

228. **Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.**

Sum the infinite series

$$\frac{1}{11.13}+\frac{1}{23.25}+\frac{1}{35.37}+\frac{1}{47.49}+\frac{1}{59.61}+\dots\dots \qquad \text{[Oxford, 1895].}$$

229. Proposed by B. F. YANNEY, Mount Union College, Alliance, O.

If $a_1{}^n+a_2{}^n+a_3{}^n+\dots\dots+a_r{}^n=A^n$, $a_1{}^m+a_2{}^m+a_3{}^m+\dots\dots+a_r{}^m>$ or $<A^m$, according as $m<$ or $>n$; provided all the letters stand for positive real numbers.

230. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Find the value of the determinant of n rows,

$$\begin{vmatrix} 5 & 2 & 0 & 0 & 0 & \dots \\ 2 & 5 & 2 & 0 & 0 & \dots \\ 0 & 2 & 5 & 2 & 0 & \dots \\ 0 & 0 & 2 & 5 & 2 & \dots \\ 0 & 0 & 0 & 2 & 5 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

[Oxford, 1900.]

231. Proposed by O. L. CALLECOT, Omaha, Neb.

Sum to infinity: $\dfrac{1}{2.3.4}+\dfrac{1}{5.6.7}+\dfrac{1}{8.9.10}+\dots\dots$

232. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If one person out of 50 die annually and one person out of 30 is born annually, how long at this rate would be required for the population to treble itself?

233. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

At what time between 10 and 11 o'clock is the second hand of a clock one minute space nearer to the hour hand than it is to the minute hand?

GEOMETRY.

254. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the cartesian equation to a line that is both tangent and normal to the cardioid.

255. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the envelope of the straight line that connects the extremities of two conjugate diameters of an ellipse.

256. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

The bisectors of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.

CALCULUS.

194. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Show that the volume of the solid generated by the re
ment of a circle, less than a semi-circle, about the diameter par
is equal to that of a sphere having a diameter equal to the chord; and hence that the volume is independent of the magnitude of the original circle, the length of the chord being known.

195. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, O.

Given a right cone of altitude h and radius r, to locate the plane parallel to its side which bisects the cone.

MECHANICS.

175. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics. Oklahoma Agricultural College, Stillwater, Oklahoma.

A cylinder descends down a plane, the inclination of which to the horizon is α, unwrapping a fine string fixed at the highest point of the plane. Find the angle through which the plane must be depressed in order that a sphere, descending under like circumstances, may experience the same acceleration.

176. Proposed by A. H. HOLMES, Brunswick, Me.

A solid cube weighs 300 pounds. If a power is applied at an angle of 45° at an upper edge of the cube, how many foot-pounds will be required to overturn the cube?

DIOPHANTINE ANALYSIS.

126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

AVERAGE AND PROBABILITY.

162. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Two points are taken at random in the surface of a circle and a chord is drawn through them. Find the average area of the segment containing the center of the circle.

GROUP THEORY.

7. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Which linear substitution will transform $x_1x_2+x_3x_4+x_5x_6=0$ into $y_1^2+y_2^2+y_3^2-y_4^2-y_5y_6=0$?

MISCELLANEOUS.

147. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

If P be a point within the scalene triangle, such that $\angle PAB=\angle PBC=\angle PCA=\psi$, then $\cot\psi=\cot A+\cot B+\cot C$..........(1), and $\operatorname{cosec}^2\psi=\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C$.........(2).

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

A list of one hundred mathematical models, made and for sale by Mr. R. P. Baker, 5519 Monroe Street, Chicago, Ill., has recently been issued. The models relate to solid geometry, linkages, crystallography, twisted cubics, cubic cones, scrolls, surfaces of the second order, etc. In view of the numerous orders received, Mr. Baker expects to devote his entire attention to the construction of models. D.

F. Strobel of Jena, has compiled a directory of all living mathematicians, physicists, astronomers, and chemists. It will be published by the firm of J. A. Barth of Leipzig, and revised every two years. S.

Mr. J. R. Hogan and Mr. E. Whitford have been appointed tutors in mathematics at the College of the City of New York. S.

The medal of the Royal Society of London was awarded to Professor W. Burnside for his researches on the theory of groups. S.

BOOKS.

A College Algebra. Seventh Edition. By J. M. Taylor, A. M., LL. D., Professor of Mathematics in Colgate University. Boston and Chicago: Allyn and Bacon. 363 pages.

To the introductory work, covering the ground of a high school course, the author devotes the first hundred pages, the remainder of the book being devoted to subjects adapted to the first year at college. In Chapter XII the fundamental notion of *functionality* is introduced and briefly illustrated by means of simple examples. In this chapter the theory of limits is also developed.

One of the chief merits of the book consists, in the opinion of the reviewer, of the introduction of the chapter on the derivatives of algebraic functions. The chapter on the development of functions in series, on convergency and divergency, logarithms and theory

of equations, are written in Dr. Taylor's inimitable style. Chapter XVII on compound interest and annuities, however, treats the latter subject in the brief manner of most algebras, the annuities there considered are *annuities certain* and not *contingent annuities based upon a mortality table*. S. E.

The Essentials of Algebra. For Secondary Schools. By Robert J. Aley, Ph. D., and David R. Rothrock, Ph. D., Professors in the University of Indiana. Silver, Burdett & Co. 1904. 295 + vii pages.

It was to be expected that as soon as the laboratory method of teaching mathematics had been sufficiently developed, text-books adapted to this form of instruction would make their appearance. The present book is the first of this kind, and is exceedingly well adapted to laboratory courses in secondary schools. As might be expected under the circumstances, the striking feature is the concreteness with which the subject is treated, principally through the chapters on graphic methods. Some of the most commendable characteristics of the book are the frequency with which diagrams are introduced, the explanation of Pascal's Triangle in connection with the binomial theorem, and Argand's representation of $i=\surd(-1)$.

The value of the book is enhanced and the pages rendered attractive to the eye by an excellent index, illustrative solutions of problems, and the frequent use of three different kinds of type.

In the opinion of the reviewer, the words "variable" and "constant" (p. 15, *et seq*) in the sense used are unfortunate; the words "unknown" and "parameter" being more suitable for the purpose. As the text explains (p. 205) $i=\surd(-1)$ may be interpreted as the unit on the axis at right angles to the axis of reals. Therefore, the term "imaginary" while sanctioned by usage and history, is undesirable. i is best regarded as the special complex number $a+bi$, where $a=0$, $b=1$.

The authors enunciate without proof the theorem that the graph of a linear equation in two variables is a straight line,—probably with the idea that this proof should be delayed to a later period in the course, and that the young student feels intuitively convinced of their truth after having constructed the graphs of several such equations.

We are indebted to the authors for an excellent text-book which combines the merits of the older texts with the recent advances in the pedagogy of mathematics.

Chicago, *March, 1905.* Alma E. Klunder.

Elements of Mechanics. Forty Lessons for Beginners in Engineering. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. 12mo, 172 pages, 142 figures. Cloth, $1.00 net. New York: John Wiley & Sons.

The aim of this volume is the application of the best methods of applied mechanics to the development of the fundamental principles and methods of rational mechanics.

"To this end, constant appeals are made to experience, by which alone the laws of mechanics can be established, numerous numerical problems are stated as exercises for the student, and a system of units is employed with which every boy is acquainted." *Preface.* The book is one that will be useful in establishing the fundamental principles of theoretical and practical mechanics. B. F. F.

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NOTE ON GROUPS OF ORDER* p^2q^2.

By O. E. GLENN, The University of Pennsylvania.

The purpose of this note is to prove the following theorem:

If a group G of order p^2q^2 $(p>q)$ has five distinct series of composition, all arrangements of the composition factors excepting (q, p, p, q) being posssble, then must $q=2$ and $p=3$. The only existent group is thus of order 36, *and it is the direct product of the tetrahedron-group and a cyclic group of order* 3.

The invariant subgroup Hq^2p is Abelian, and Hp^2q is a divisible type $\{S_1, S_3\}\{S_2\}$ defined by

$$S_1{}^p=S_2{}^p=S_3{}^q=1,\quad S_1S_2=S_2S_1,\quad S_3S_2=S_2S_3,\quad S_3'S_1S_3=S_1{}^a.$$

The Sylow subgroup $I_{q^2}=\{S_3, S_4\}$ is invariant under H_{q^2p} and hence under G. Likewise $\{S_1, S_2\}$ is self-conjugate in G, and these two subgroups contain all the subgroups of G of orders q and p, respectively. The operator S_1 transforms $\{S_3\}$ into p conjugates within $\{S_3, S_4\}$, and since the number of these cannot exceed

$$N_q=\frac{q^2-1}{q-1}=q+1$$

it follows that $p=q+1$, and therefore $p=3$, $q=2$.

Hence $a\equiv -1$ (mod 3), and the $N_2=3$ subgroups are

$$\{S_3\},\quad \{S_4\},\ \text{and}\ \{S_3S_4\}.$$

*Le Vavavasseur has given in Comptes Rendus, Vol. 128 (1899), p. 152, a list of the groups of order p^2q^2, the proofs having been suppressed. [ED. D.]

If these be transformed by S_2, the number, ρ, remaining invariant, satisfies the congruence $3-\rho\equiv 0 \pmod 3$, and since $S_2^{-1}\{S_3\}S_2=\{S_3\}$, we have $\rho=3$, and $S_2S_4=S_4S_2$. Suppose next that $S_4^{-1}S_1S_4=S_1^xS_2^y$.

$$(S_3S_4)^{-1}S_1(S_3S_4)=S_1^{-x}S_2^{-y}=(S_4S_3)^{-1}S_1(S_4S_3)=S_1^{-x}S_2$$
$$-2y\equiv 0 \pmod 3);\ y\equiv 0 \pmod 3),$$

and as S_4 is not permutable with $S_1 x\equiv -1 \pmod 3$. So that $G=\{S_3, S_4, S_1\}\{S_2\}$ is defined by the relations

$$S_1^3=S_2^3=S_3^2=1,\quad S_1S_2=S_2S_1,\quad S_3S_4=S_4S_3,\quad S_3^{-1}S_1S_3=S_1^{-1},\quad S_2S_3=S_3S_2,$$
$$S_4^{-1}S_1S_4=S_1^{-1},\quad S_2S_4=S_4S_2.$$

The subgroups $\{S_3, S_4, S_1\}$ will be recognized as the abstract form of the tetrahedron rotation group.

TESTS OF DIVISIBILITY BY 7, 13, AND 17.

By MISS ALICE CHURCH, New York City.

1. Seven is an exact divisor of any number of which the units figure doubled differs from the number represented by the remaining digit or digits by zero or by a multiple of seven. Thus, in the number 14, twice 4 are 8; 1 from 8 leaves 7.

In 168, twice 8 are 16; 16 from 16 leaves 0.

In 532, twice 2 are 4; 4 from 53 leaves 49.

Corollary A. When the remainder after subtraction is 0, the original number is a multiple of 3 as well as of 7, therefore is a multiple of 21.

When the number to be treated is so large that the remainder found by subtracting the doubled unit from the rest of the number is too large to be judged by inspection, the same test may be applied to the remainder as to the original number. This process may be repeated until a remainder be found which is small enough to be factored by inspection.

For example, 22134; twice 4 are eight; 8 subtracted from 2213 leaves 2205; twice 5 are 10; 10 from 220 leaves 210; 210 at once appears as a multiple of 7.

Demonstration. Multiplying the units figure by 2 and placing the product in the tens column and ignoring the units figure in the subtraction is really multiplying the units figure by 21, which is a multiple of 7. The test, then, becomes merely the subtraction of a multiple of seven from a possible multiple of 7, or vice versa, and as the difference between multiples must be a multiple, the number tested is divisible by 7 if the difference found is a multiple of 7.

The value of the test rests upon the facility with which the multiple of 21 is created, inasmuch as to multiply by 2 is a mental process much more surely within the mind of a child than dividing by 7.

2. A test for the divisibility by 13 is to be found by a similar process and upon the same principles—being based on the fact that 91 is a multiple of 7. Multiply the unit figure by 9 and find the difference between the product and the number without its unit figure. Thus, in 1183, $9\times3=27$, $118-27=91$, and in 91, $9\times1=9$, $9-9=0$. For 325, $9\times5=45$, $45-32=13$.

3. Likewise for 17, multiply by 5, since 51 is a multiple of 17. So for 595, $5\times5=25$; $59-24=34$. For 2244, $5\times4=20$, $224-20=204$; $5\times4=20$, $20-20=0$.

NOTE ON THE EVOLUTE OF AN ALGEBRAIC CURVE.

By A. H. WILSON, Instructor of Mathematics, University of Illinois.

The following method of forming the evolute of an algebraic curve may be of interest.

Let $f(x, y)=\varphi$ represent the curve, and $y-y_1=l(x-x_1)$ its normal at the point (x_1y_1) on the curve, l being a function of x_1 and y_1. The elimination of x_1 (or y_1) between $f(x_1, y_1)=0$ and $\beta-y_1=l(\alpha-x_1)$, gives an equation

$$\varphi(y_1)=0 \text{ (or } \psi(x_1)=0),$$

whose roots are the ordinates (or the abscissas) of the points on the curve the normals at which pass through the point (α, β).

The evolute may be regarded as the locus of points from which two of the normals through (α, β) to the curve are coincident; and hence the equation of the evolute is the relation between α and β obtained by setting equal to zero the discriminant of $\varphi=0$ (or $\psi=0$).

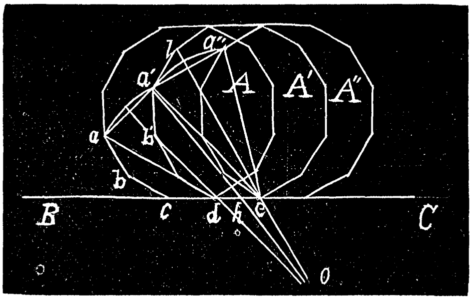
The application of the method is obviously very limited.

DETERMINATION OF THE RADIUS OF CURVATURE OF THE CYCLOID WITHOUT THE AID OF THE CALCULUS.

By FREDERIC R. HONEY, Hartford, Conn.

Let A represent any regular polygon. If we roll it along the straight line BC into the positions A', A'', ... bringing each side in succession into coinci-

dence with it, any angular point as a, will trace a series of arcs of circles aa', $a'a''$, ... whose centers will be on the line BC. The distance between two consecutive centers will be equal to the side of the polygon. The arc aa' will be described with the radius da; $a'a''$ with the radius ea', ...



The center o of an arc $aa'a''$ which will pass through the points a, a', a'', will be at the intersection of the bisector of the angle ada' and the bisector of the angle $a'ea''$. Since the polygon is inscriptible, the angle $a'el$ is equal to the alternate angle $ea'd$. Therefore, leo is parallel to $a'd$. Similarly, $b'do$ is parallel to $a'e$. Therefore, $a'eod$ is a parallelogram. Its diagonal oa', the radius with which the arc $aa'a''$ is described, is bisected at h.

The above demonstration is applicable to a regular polygon with *any* number of sides. We will now suppose that the number is increased. The length of the side diminishes, and the points d, h, and e approach each other. At the limit, when the polygon becomes a circle, they coalesce, and ha' is the normal. The broken curved line $aa'a''$.......... becomes a cycloid, and the radius of curvature $oa'=2ha'$.

REMARKS ON DEFINITIONS IN TEXT-BOOKS ON GEOMETRY.

By G. W. GREENWOOD, M. A., McKendree College.

What is a circle? A common definition is that it is a plane figure—or a portion of a plane—bounded by a curved line, every point of which is equally distant from a point *within*, called the center. This gives us the impression that a circle is a disc, whereas in more advanced work it is regarded, with other conic sections, merely as a plane curve. It would be better if it were so defined in elementary texts. When we turn, however, to some texts which define it thus, we find something like this: 'A circle is a plane *closed line*, such that all straight lines joining *any* point on this line to a *certain point within* the figure, are equal.'

I shall endeavor to show that such a definition, like the more common one first given, is by no means logical. For this purpose, let us compare them with the following, which, while not altogether free from objections, is, I believe, logical: The locus of points in a plane, at congruent distances from a fixed point in the plane, is a circle. It will be noticed here that nothing is stated or implied concerning the form of the locus or any other properties save the one stated. The word *within* in the earlier definition is unnecessary unless, like a vermiform appendix, it indicates the evolution of this definition from the one first

quoted. But in addition to this we see that the definition assumes that the circle divides its plane into at least two parts, one of which is *within*. We are then supplied with other information, such as, that *since* a circle is a *closed curve*, a line from any point *within* to any point *without* the circle *intersects* the circle at least once. This assumes, among other things, that the locus is continuous; yet how all these conclusions are to be deduced from the definition, it would be difficult to state.

Most of the preliminary data in the case of circles, as given in current texts, is based entirely on our visual perception of the figure, with no attempt at deducing it from the definition; none of their off-hand statements are more evident at a glance than the fact that a straight line cannot "intersect" a circle in more than two points, yet a capricious sense of logical rigor requires that this assertion be demonstrated at length.

The premature definition of secants, tangents, tangent circles, etc., in current texts, tends to dull the sense of logical order.

We are not surprised, when the definition given of a circle is similar to one of the two quoted, to find many fallacies in their treatment of these figures. For example, consider the following: 'The line joining a point to the center of a circle is less than, or greater than, a radius, according as the point is within, or without, the circle. Let P be a point within, and Q a point without, a circle whose center is O. *Extend* OP to *meet* the circle in A, and *let* OQ *cut* the circle in C.' These statements are the very facts we wish to prove; namely, that C lies between O and Q, and that A does not lie between O and P.

Generally, with no consideration of the relative magnitudes of arcs of a circle, we are told that a chord subtends two arcs, of which the *lesser* is meant unless otherwise stated. Most of the subject as usually treated is equally vulnerable.

ON KINEMATIC GEOMETRY.—A NEW INVERSOR.

By JOHN J. QUINN, Warren, Pa.

The Theory of Inversion, an important method of investigation especially in many problems of mathematical physics, is one of the distinctive features of modern geometry. It is a method of transformation whereby one surface, or curve, is represented by another surface or curve in a manner that insures similarity in the most minute details. In general, the inverse of a sphere is a sphere; and the inverse of a circle is a circle. In order that two points be in inverse relation it is necessary that they fulfil two conditions:

1. *They must be always collinear with a fixed point.*
2. *The product of their distances from the fixed point must be constant.*

By harnessing this principle it is possible to draw a line that is mathe-

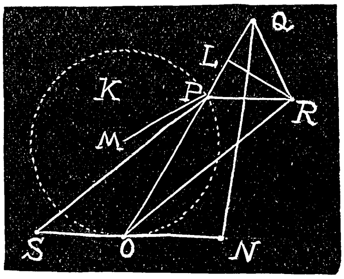
matically straight. This can be done by a linkwork so constructed that when the whole system is moved there will be three points, as O, P, and Q collinear in every position, and in the relation $OP\times OQ$=constant.

A linkwork that will describe the inverse of a given curve or surface is termed an Inversor.

The inverse points as O, P, and Q, in an inversion are called the foci.

THEOREM. *Let the side OS of the parallelogram $ORPS$ be extended to N, making ON equal to OS, and located so that NQ equals OR, and RQ equals PR. Then if this figure be embodied in a linkage:*

1. *The points O, P, and Q, are always collinear.*
2. *The points O, P, and Q, are inverse points.*
3. *If the focus O of the linkage be pivoted to a fixed base, and the focus P be constrained to move in a circle through O, the focus Q will describe a straight line.*



Given: $SP=OR=NQ$; $SO=ON=PR=RQ$.

Proof 1. $\triangle ONR \cong \triangle QNR$.

But $OP \parallel NR$.

$\therefore$ The points O, P, and Q are collinear.

Proof 2. Draw RL perpendicular to OPQ. Then $OP=OL-LP=\sqrt{(OR^2-RL^2)}-\sqrt{(PR^2-RL^2)}$; $OQ=OL+LP=\sqrt{(OR^2-RL^2)}+\sqrt{(PR^2-RL^2)}$.

$\therefore OP.OQ=OR^2-PR^2$=a constant.

$\therefore$ The points O, P, and Q are inverse points.

Proof 3. $\because$ O, P, and Q are inverse points, and since a straight line is the inverse of a circle through the pole of inversion,

$\therefore$ The line QV is a straight line, the inverse of the circle K. Q. E. D.

SIX PROPOSITIONS ON PRIME NUMBERS.

By R. D. CARMICHAEL, Hartselle, Ala.

PROPOSITION A. *If p is an odd prime,* $(1.2.3\ldots\frac{p-1}{2})^2+(-1)^{\frac{1}{2}(p-1)}\equiv 0$ (*mod* p).

By Wilson's theorem, if p is prime

$$1.2.3\ldots(p-1)+1\equiv 0 \pmod p \qquad (1).$$

This may be written

$$(1.2.3\ldots\frac{p-3}{2}.\frac{p-1}{2})(p-\frac{p-1}{2})(p-\frac{p-3}{2})\ldots(p-2)(p-1)+1\equiv 0 \pmod p \qquad (2).$$

Expanding and casting out all terms containing p as a factor (p being an odd prime),

$$(1.2.3.....\tfrac{p-1}{2})^2+(-1)^{\frac{1}{2}(p-1)}\equiv 0 \pmod p \qquad (3).$$

PROPOSITION B. *If* $(1.2.3.....\frac{p-1}{2})^2+(-1)^{\frac{1}{2}(p-1)}\equiv 0$ *(mod p), p is prime.*

It is evident that we may revert from (3) to (2) by re-inserting the cast out terms containing p as a factor. (2) is but another form of (1). Hence, we have only to verify the well-known theorem that p is a prime, if

$$1.2.3.....(p-1)+1\equiv 0 \pmod p \qquad (4)$$

For any value of p *not* prime, except $p=4$, it is easily shown that

$$1.2.3.....(p-1)\equiv 0 \pmod p \qquad (5).$$

Hence, when (4) is true, p is prime.

PROPOSITION C. *If* $4n+1$ *is prime, it may be expressed as the sum of two parts r and s such that* $r^2+1\equiv 0$, $s^2+1\equiv 0$, *and* $rs-1\equiv 0$ *(mod p).*

For p a prime of the form $4n+1$, (3) may be written

$$(1.2.3.....2n)^2+1\equiv 0 \pmod{4n+1} \qquad (6).$$

This may be written in either of the forms,

$$[l(4n+1)+r]^2+1\equiv 0 \pmod{4n+1} \qquad (7),$$
$$[(l+1)(4n+1)-s]^2+1\equiv 0 \pmod{4n+1} \qquad (8),$$

where l is an integer and $r+s=4n+1$. Hence, expanding and casting out the terms containing $4n+1$ as a factor, we get

$$r^2+1\equiv 0,\ s^2+1\equiv 0 \pmod{4n+1} \qquad (9),$$

from which easily follsws $rs-1\equiv 0$.

PROPOSITION D. *If* $a+1$ *and* $2a+1$ *are both primes,* $(1.2.3.....a)^4-1\equiv 0$ [*mod* $(a+1)(2a+1)$].

By Wilson's theorem,

$$1.2.3....a+1\equiv 0 \pmod{a+1} \qquad (10).$$
$$\therefore (1.2.3.....a)^4-1\equiv 0 \pmod{a+1} \qquad (11).$$

By Proposition A above,

$$(1.2.3.....a)^2+(-1)^a \equiv 0 \pmod{2a+1} \qquad (12).$$

$$\therefore (1.2.3.....a)^4-1 \equiv 0 \pmod{2a+1} \qquad (13).$$

Since $a+1$ and $2a+1$ are prime to each other, (11) and (13) lead to the theorem stated.

PROPOSITION E. *If* $(1.2.3.....a)^4 \equiv 0$ [*mod* $(a+1)(2a+1)$], $a+1$ and $2a+1$ *are both prime.*

We have to show that $a+1$ and $2a+1$ are prime when (11) and (13) hold.

If $a+1=4$, (11) does not hold. For all other values of $a+1$ *not* prime, $1.2.3.....a \equiv 0 \pmod{a+1}$. Hence, when $a+1$ is not prime, (11) does not hold. Therefore, $a+1$ is prime under the given condition.

If $2a+1$ is *not* prime, $1.2.3.....2a \equiv 0 \pmod{2a+1}$.

$$\therefore (1.2.3.....(a-1)a.[(2a+1)-a][(2a+1)-(a+1)].....$$
$$.....[(2a+1)-2][(2a+1)-1] \equiv 0 \pmod{2a+1}.$$

Expanding and casting out terms containing $2a+1$ as a factor, we get

$$(1.2.3.....a)^2 \equiv 0 \pmod{2a+1} \qquad (14),$$

if $2a+1$ is *not* prime. Hence, when (13) holds, as (14) does not then hold, $2a+1$ is odd.

PROPOSITION F. *If* $a+1$ *and* $2a+1$ *are both prime,* $(a+1)(2a+1)$ *may be expressed as the sum of* r *and* s *such that* $r^4-1 \equiv 0$, $s^4-1 \equiv 0$, *and* $r^2s^2-1 \equiv 0$ [*mod* $(a+1)(2a+1)$].

The demonstration depends upon Proposition D above, and is similar in method to that of Proposition C. It may easily be supplied by the reader.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

No. 223 was also solved by L. E. Newcomb.

Mr. L. S. Shively calls attention to the fact that a solution of No. 225 is given in C. Smith's "A Treatise on Algebra," page 183, Ex. 4.

226. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Find the real roots of the system

$$x^2+w^2+v^2=a^2, \quad vw+u(y+z)=bc,$$
$$w^2+y^2+u^2=b^2, \quad wu+v(z+x)=ca,$$
$$v^2+u^2+z^2=c^2, \quad uv+w(x+y)=ab.$$

Solution by A. H. HOLMES, Brunswick, Me.

$$x^2+w^2+v^2=a^2\text{.........}(1),\quad vw+u(y+z)=bc\text{.........}(4),$$
$$w^2+y^2+u^2=b^2\text{.........}(2),\quad wu+v(z+x)=ca\text{.........}(5),$$
$$v^2+u^2+z^2=a^2\text{.........}(3),\quad uv+w(x+y)=ab\text{.........}(6).$$

From (2), (3), and (4),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[vw+u(y+z)]^2.$$
$$\therefore (yv-uw)^2+(uv-zw)^2+(u^2-yz)^2=0.$$
$$\therefore yv=uw\text{.........}(7);\ uv=zw\text{.........}(8);\ u^2=yz\text{.........}(9).$$

From (1), (3), and (5),

$$(x^2+w^2+v^2)(v^2+u^2+z^2)=[wu+v(z+x)]^2.$$
$$\therefore v^2=xz\text{.........}(10);\ vw=xu\text{.........}(11).$$

From (2), (3), and (6),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[uv+w(x+y)]^2.$$
$$\therefore w^2=xy\text{.........}(12).$$

Substituting xy and xz for w^2 and v^2 in (1), xy and yz for w^2 and u^2 in (2), and xz and yz for v^2 and u^2 in (3), adding the three equations and extracting square root, we obtain $x+y+z=\sqrt{(a^2+b^2+c^2)}$.

Substituting xu for vw in (4), $u=\dfrac{bc}{\sqrt{(a^2+b^2+c^2)}}$.

Similarly, $v=\dfrac{ac}{\sqrt{(a^2+b^2+c^2)}}$, $w=\dfrac{ab}{\sqrt{(a^2+b^2+c^2)}}$,

$$x=\frac{a^2}{\sqrt{(a^2+b^2+c^2)}},\ y=\frac{b^2}{\sqrt{(a^2+b^2+c^2)}},\ z=\frac{c^2}{\sqrt{(a^2+b^2+c^2)}}.$$

227. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Solve $x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3=11$; $x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^2y^3+2x^2y^4+4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4+x^2y^5+x^3y^5=30$.

Solution by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Put $X=(x+y+xy+x^2y+xy^2)$, and $Y=(x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3)$; then the given equations become, respectively,

$$X+Y=11\text{.........}(\alpha),\ \text{and}$$
$$XY=30\text{.........}(\beta).$$

$\therefore X=6$, or 5; and $Y=5$, or 6.

By putting $X_1=(x+y+xy)$, and $Y_1=(x^2y+xy^2)$, the expressions represented by X and Y give, respectively,

$$X_1+Y_1=6, \text{ or } 5 \ldots\ldots(\alpha_1), \text{ and}$$
$$X_1Y_1=5, \text{ or } 6 \ldots\ldots(\beta_1).$$

$\therefore X_1=5$, 1, 3, or 2; and $Y_1=1$, 5, 2, or 3.

By putting $X_2=(x+y)$ and $Y_2=(xy)$, the expressions represented by X_1 and Y_1 give, respectively,

$$X_2+Y_2=5, 1, 3, \text{ or } 2 \ldots\ldots(\alpha_2), \text{ and}$$
$$X_2Y_2=1, 5, 2, \text{ or } 3 \ldots\ldots(\beta_2).$$

$$\therefore x+y=\tfrac{1}{2}(5\pm\sqrt{21}),\ \tfrac{1}{2}(1\pm\sqrt{-19}),\ 2 \text{ or } 1, \text{ or } 1\pm\sqrt{-2}; \text{ and}$$
$$xy=\tfrac{1}{2}(5\mp\sqrt{21}),\ \tfrac{1}{2}(1\mp\sqrt{-19}),\ 1 \text{ or } 2, \text{ or } 1\mp\sqrt{-2}.$$

Solving these *eight* simultaneous equations, we have the *sixteen* values of x and y.

Also solved by J. Scheffer, Henry Heaton, G. B. M. Zerr, and A. H. Holmes.

GEOMETRY.

254. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the cartesian equation to a line that is both tangent and normal to the cardioid.

[No solution of this problem has been received.]

255. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the envelope of the straight line that connects the extremities of two conjugate diameters of an ellipse.

I. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

If we project the ellipse into a circle, the conjugate diameters are projected into perpendicular diameters of the circle, whose chord envelopes a concentric circle. Hence in the original figure the chord envelopes a similar, and similarly situated concentric ellipse.

II. Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

The equation to the line is, with usual notation,

$$\frac{x}{a}\cos(\varphi+\tfrac{1}{4}\pi)+\frac{y}{b}\sin(\varphi+\tfrac{1}{4}\pi)=\cos\tfrac{1}{4}\pi, \text{ or } \left(\frac{x}{a}+\frac{y}{b}\right)\cos\varphi-\left(\frac{x}{a}-\frac{y}{b}\right)\sin\varphi=1,$$

$$i.e., \left(\frac{x}{a}+\frac{y}{b}-1\right)-2\left(\frac{x}{a}-\frac{y}{b}\right)\tan\tfrac{1}{2}\varphi-\left(\frac{x}{a}+\frac{y}{b}+1\right)\tan^2\tfrac{1}{2}\varphi=0.$$

$\therefore$ The equation to the envelope is

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2+\left(\frac{x}{a}+\frac{y}{b}+1\right)\left(\frac{x}{a}+\frac{y}{b}-1\right)=0, \textit{ i. e., } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\tfrac{1}{2}.$$

Solved similarly by J. Scheffer.

256. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

The bisectors of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.

Solution by G. W. GREENWOOD, M. A.. McKendree College. Lebanon, Ill.; FREDERIC R. HONEY, Ph. D., Trinity College, Hartford, Conn.; R. D. CARMICHAEL, Hartselle, Ala., and A. H. HOLMES, Brunswick, Me.

Denote the quadrilateral by $ABCD$, and the bisectors of the angles A, B, C, D by a, b, c, d, respectively. Suppose AB, DC intersect at O; a, d at X; and b, c at Y. We can easily show that, if B lies between A and O,

$$\angle AXD=\tfrac{1}{2}\pi+\tfrac{1}{2}\angle AOD;$$
$$\angle BYC=\tfrac{1}{2}\pi-\tfrac{1}{2}\angle AOD.$$

Hence, a pair of opposite angles of the quadrilateral whose sides are, consecutively, a, b, c, d, are supplementary, and it is therefore cyclic.

*** The problem admits of the following interesting extension: If a', b', c', d' are the bisectors, respectively, of the exterior angles A, B, C, D, the six quadrilaterals whose consecutive sides are, respectively, $(abcd)$, $(abc'd')$, $(bcd'a')$, $(cda'b')$, $(dab'c')$, $(a'b'c'd')$, are cyclic. It is necessary that no two sides of the given figure be parallel. GREENWOOD.

Also solved by F. P. Matz, J. Scheffer, G. B. M. Zerr, and W. J. Greenstreet. A. H. Holmes also solved No. 252.

CALCULUS.

194. Proposed by G. W. GREENWOOD. M. A. (Oxon), Lebanon, Ill.

Show that the volume of the solid generated by the revolution of a segment of a circle, less than a semi-circle, about the diameter parallel to its chord, is equal to that of a sphere having a diameter equal to the chord; and hence that the volume is independent of the magnitude of the original circle, the length of the chord being known.

Solution by W. L. TRYON. Ithaca. N. Y.; R. D. CARMICHAEL, Hartselle, Ala.; and W. J. GREENSTREET, Editor of The Mathematical Gazette, Stroud, England.

Let the chord be of length $2b$ in circle of radius a; and take as axes the parallel diameter and the perpendicular bisector of the chord. The distance of the chord from the center is $\sqrt{(a^2-b^2)}$. The required volume is

$$V=2\int_0^b (a^2-x^2)dx-2\pi b(a^2-b^2)=2\pi b(a^2-\tfrac{1}{3}b^2)-2\pi b(a^2-b^2)=\tfrac{4}{3}\pi b^3.$$

This is independent of a, and is equal to the volume of a sphere of diameter $2b$.

Also solved by C. Hornung, J. Scheffer, A. H. Holmes, and the Proposer.

MECHANICS.

173. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A squirrel is in a cylindrical cage and oscillating with it about its axis which is horizontal. At the instant when he is at the highest point of the oscillation, he leaps to the opposite extremity of the diameter and arrives there at the same instant as the point at which he left. Determine his leap completely.

[No satisfactory solution has been received.]

174. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

By what per cent. is the striking force of a hailstone increased in falling 1000 feet through a stratum of atmosphere moving uniformly eastward at the rate of 60 miles an hour?

Solution by the PROPOSER.

Let S_1=the *striking-force* of the hail-stone falling 1000 feet, and $S_2=mS_1$ =that caused by the horizontal motion of the atmosphere during the time required to fall 1000 feet; then the resultant striking-force is $\sqrt{(1+m^2)}S_1$. Assuming a mass of ice suitable for a hail-stone, we can easily calculate m.

The required percentage of increase in striking-force becomes

$$I=\left(\frac{\sqrt{(1+m^2)}-1}{1}\right) \text{ of } 100\%.$$

175. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics. Oklahoma Agricultural College, Stillwater, Oklahoma.

A cylinder descends down a plane, the inclination of which to the horizon is α, unwrapping a fine string fixed at the highest point of the plane. Find the angle through which the plane must be depressed in order that a sphere, descending under like circumstances, may experience the same acceleration.

Solution by G. W. GREENWOOD, M. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let ω be the angular velocity, and v the velocity parallel to the plane, after rolling a distance l from the highest point. Equating the *vis viva* of the body to twice the work done by gravity, we get, in the case of the cylinder,

$$\tfrac{1}{2}ma^2\omega^2+mv^2=2mgl\sin\alpha,$$

where a is the radius of the cylinder.

$\therefore 3v^2=4gl\sin\alpha$, since $a\omega=v$.

In the case of the sphere we get

$$\tfrac{2}{5}m'a'\omega^2+m'v^2=2mgl\sin(\alpha-\theta),\ i.\ e.,\ 7v^2=10gl\sin(\alpha-\theta),$$

where θ is the amount of depression of the plane.

$\therefore 14\sin\alpha=15\sin(\alpha-\theta)$, or $\theta=\alpha-\sin^{-1}(\frac{14}{15}\sin\alpha)$.

DIOPHANTINE ANALYSIS.

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find (1) three square numbers whose sum is a cube; (2) three cube numbers whose sum is a square.

I. Remark by J. SCHEFFER, Hagerstown, Md.

(1) The sum of the three squares 9, 16, 100 gives the cube number 125, hence $9n^6+16n^6+100n^6=125n^6=(5n^2)^3$, where n=any integer. There are doubtless other sets of numbers satisfying this condition.

(2) Since the sum of the cubes of three Pythagorean numbers is a square, one set of numbers satisfying this condition is: $27\times6^{6n-3}$, $64\times6^{6n-3}$, $125\times6^{6n-3}$, whose sum$=6^{6n}$.

II. Remark by the PROPOSER.

(1) Since $5^2+6^2+8^2=5^3$, the numbers $5n$, $6n$, $8n$, n=integer, satisfy the required condition.

(2) Since $1^3+2^3+3^3=6^2$ the numbers n^3, $8n^3$, $27n^3$, n=integer, satisfy the required condition.

Also solved by G. B. M. Zerr, and A. H. Holmes.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

What *values* of x will make $E=\dfrac{(x+7)(x+5)}{(x-7)(x-5)}$ represent square numbers?

Solution by G. B. M. ZERR.

$$E=\frac{(x+7)(x+5)}{(x-7)(x-5)}=\frac{(1+7/x)(1+5/x)}{(1-7/x)(1-5/x)}.$$

Hence, two values of x which satisfy the required condition are 0, ∞.

Also solved by J. Scheffer, and the Proposer.

AVERAGE AND PROBABILITY.

158. Proposed by F. P. MATZ. Sc. D., Ph. D., Reading, Pa.

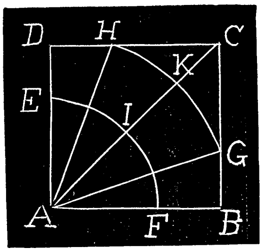
In a given square an arc is described at random the center being one of the vertices of the square. What is the probability that this arc is longer than a side of the square?

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the square; A the vertex from which the random arc is drawn; and EIF and HKG arcs whose lengths are equal to a, the length of a side of the square. Then if the random arc falls between the arcs EIF and HKG its length will exceed the length of a side of the square.

Since no law of distribution of the events is given, we will assume that the intersections of the arcs with the line AC are uniformly distributed on AC.

Hence, if the random arc intersects AC between I and K its length will exceed a, the length of a side of the square.



Hence, the required probability is

$$p=\frac{IK}{AC}=\frac{AK-AI}{AC}.$$

But, since $EIF=\frac{1}{2}\pi.AI=a$, $AI=2a/\pi$, and likewise, $AK=a/\theta$, where $\theta=\angle HAG$.

$\therefore p=\dfrac{(a/\theta-2a/\pi)}{a\sqrt{2}}=\dfrac{1}{\theta\sqrt{2}}-\dfrac{\sqrt{2}}{\pi}$, where θ may be found by the method of Double Position from the equation, $\cos(\frac{1}{4}\pi-\frac{1}{2}\theta)=\theta$.

For $\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=\dfrac{AB}{AG}=\dfrac{a}{AG}$, and $\theta.AG=HKG=a$.

Hence, $\theta=\dfrac{a}{AG}$, and therefore $\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=\theta$.

Solving this equation we find $\theta=.95266$ radians.

$$\therefore p=\frac{1}{.95266\sqrt{2}}-\frac{\sqrt{2}}{\pi}=.2920+.$$

Also solved by F. P. Matz who distributes the events in proportion to the area between the limiting arcs and the area of the square; G. B. M. Zerr, who gets as a result by using the calculus, .0734; and Henry Heaton, who finds the mean lengths of the arcs.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

A box contains n tickets numbered from 1 to n. How many draws, on the average, will it take to draw all the numbers, each ticket being replaced before drawing again? What is the numerical result for $n=2$ and $n=6$?

Solution by W. W. LANDIS, A. M., Professor of Mathematics, Dickinson College, Carlisle, Pa.

The chance of drawing any particular number at least once in p drawings is

$$1-\left(1-\frac{1}{n}\right)^p.$$

The chance that all will be drawn in p drawings (p being, of course $>n$) is

$$\left[1-\left(1-\frac{1}{n}\right)^p\right]^n,$$

which by the conditions of the problem must equal $\frac{1}{2}$. Solving this equation for p,

$$p=\frac{\frac{1}{n}\log 2-\log(2^{1/n}-1)}{\log n-\log(n-1)}.$$

For $n=2$, $p=1.75+$, hence two drawings must be made; for $n=6$, $p=12.15+$; hence, thirteen drawings must be made.

Also solved with the same result by F. O. Whitlock, and by a different method, which seems to me to be incorrect, by S. A. Corey, Hiteman, Iowa. The problem is equivalent to that of Problem IX, page 52, of Meyer's *Wahrscheinlichkeitsrechnung*. F.

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

The year 1905 *began*, and will *end*, on a Sunday. Prove that this can not occur again until the year 2015.

Solution by WILLIAM HOOVER, Ph. D., Athens, Ohio.

The Dominical Letter for Sunday when on January 1 is A, and also when on December 31, the year being common. Those common years in the present century fulfilling the required conditions must have A for their Dominical Letter; such years are 1905, 1911, 1922, 1933, 1939, 1950, 1961, 1967, 1978, 1989, 1995, sufficient to show that the statement in the problem is not true.

REMARK BY PROPOSER. The year 2015 will begin on a Thursday.
Also solved by A. H. Holmes, Henry Heaton, G. B. M. Zerr, and G. W. Greenwood.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If an *unknown* curve be described under a constant acceleration not tending to the center and the hodograph is a cardioid, what is the unknown curve?

I. Solution by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Let r and p be the radius vector and perpendicular upon the tangent to the curve at the outer extremity of r and r'; p' the analogous lines in the hodograph, and h the double area generated by r in a unit of time.

Then by the theory of the hodograph,

$$r'=\frac{h}{p}\ \ldots\ldots\ (1),\quad p'=\frac{h}{r}\ \ldots\ldots\ (2).$$

Also, from the theory of central forces,

$$k=\frac{h^2}{p^3}\frac{dp}{dr}\ \ldots\ldots\ (3),$$

and for the cardioid,

$$p'^2=\frac{r'^3}{2a}\ \ldots\ldots\ (4).$$

Substituting r' and p' from (1) and (2) in (4),

$$h=\frac{2ap^3}{r^2}\text{.........}(5).$$

Then (5) in (3), gives

$$kr^4dr=4a^2p^2dp\text{.........}(6),$$

the differential equation to the required orbit.

Integrating (6), and supposing r and p to vanish together,

$$\frac{k}{5}r^5=a^2p^4\text{.........}(7),$$

the required orbit.

II. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

Let the equation to the hodograph be $r=2a\cos^2\frac{1}{2}\theta$. Since the acceleration in the original curve is constant the velocity of the point in the hodograph is constant, and $s=4akt$, where k is some constant; that is $kt=\sin\frac{1}{2}\theta$.

From the equation to the hodograph we have $v=2a\cos^2\frac{1}{2}\psi$, where v is the velocity in the orbit and ψ is the inclination of the tangent at that point to the initial line.

$$\therefore \frac{ds}{dt}=2a\cos^2\tfrac{1}{2}\psi=2a(1-k^2t^2).$$

$$\therefore s=2a(t-\tfrac{1}{3}k^2t^3);\ i.\ e.,\ s=\frac{2a}{k}(\sin\tfrac{1}{2}\psi-\tfrac{1}{3}\sin^3\tfrac{1}{2}\psi),$$

which is the intrinsic equation to the orbit.

Also solved by G. B. M. Zerr, S. A. Corey, and the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

234. Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Prove that $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\text{.........}=n!$

235. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Easter Sunday, 1905, was on April 23. How often in the last one hundred years has this occurred, and when?

236. Proposed by L. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

Sum to infinity the series $\frac{n^2}{(n+1)!}$ beginning with $n=1$.

GEOMETRY.

257. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Construire un triangle équilatéral sachant qu'el doit s'appuyer par ses trois sommets sur trois circonférences concentriques données. *Rouché et Comberousse.*

258. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Mo.

Prove that the tangents to an ellipse from any external point subtend equal angles at the focus, by means of the formula tan $\phi=(m_1-m_2)\ /\ (1+m_1 m_2)$, where ϕ is the angle between the focal radius of either of the points of tangency and the line joining the focus and the external point, and m_1 and m_2 are the slopes of these two lines.

259. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Given three non-intersecting circles; to draw eight tangent circles, each tangent to all three of the given circles.

CALCULUS.

196. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

The shortest tangent intercepted by the axes of the ellipse to which the tangent is drawn, equals the sum of the semi-axes of the ellipse.

197. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

$$\int_0^\infty \frac{\sin mx \cos nx}{x} dx.$$

189. Proposed by M. E. GRABER, A. M., Heidelburg University, Tiffin, O.

Show that $e\int_1^\infty$, $\varepsilon^2\int_2^\infty$, ...$\varepsilon^n\int_n^\infty$,.......... are integers divisible by $(p+1)!$, when

the expression under the integral is $Z^p\Big[(z-1)..................(z-n)\Big]^{p+1} e^{-z}\,dZ$.

MECHANICS.

178. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

A weight W is drawn up a rough conical hill of height h and slope α, and

the path cuts all the lines of greatest slope at the constant angle β. Find the work done in attaining the summit.

[Problem 11, page 226, *Johnson's Theoritical Mechanics.*]

179. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the *velocity* of a body moving under an acceleration tending to the center *varies* as the radius of curvature, the body will describe a cycloid.

MISCELLANEOUS.

148. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Given $\sin 3\phi+\cos 3\phi=m$............(1), and $\cos\phi-\sin\phi=x$..............(2), to find x extremes of m.

AVERAGE AND PROBABILITY.

163. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle.

164. Proposed bv J. O. Mahoney, B. E., M. Sc., Central High School, Dallas, Texas.

If m is prime, and the numbers 0, 1, 2, 3,...................., m^2-1 are placed at random in the form of a square, the probability that the square is *hyper-magic* is

$$\frac{(m-1)\,m}{(m^2-2)\,!}$$

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

The Chicago Section of the American Mathematical Society met in Chicago on April 29. E.

Mr. Newton Ensign, of McKendree College, a student of our well known contributor Prof. G. W. Greenwood, was awarded the Rhodes Scholarship for Illinois. He will pursue the honor mathematical course at Oxford University. E.

The Open Court Publishing Co., of Chicago, has just issued a portfolio of twelve portraits of eminent mathematicians, edited by Professor David Eugene Smith. It includes the portraits of DeCartes, Pythagoras, Archimedes, Fermat,

Leonardo of Pisa, Euclid, Leibnitz, Napier, Vieta, Newton, Thales. The portraits are printed by a photographic process, and are issued in two forms, the first on Japanese paper, the price of which is $5, and the second on plate paper, at $3. The originals are from Professor Smith's large collection. With each portrait is a biographical note introducing a brief biography. E.

Editor Finkel has been appointed to a Harrison Fellowship in the University of Pennsylvania for the year 1905-1906. The Board of Trustees of Drury College has granted him a year's leave of absence and he will spend the coming year in study in Philadelphia. Mr. O. E. Glenn, Fellow in Mathematics in the University of Pennsylvania, has been appointed Acting Professor of Mathematics and Physics during his absence.

Dr. Edward Kasner has been promoted to an instructorship in Mathematics in Barnard College, Columbia University.

Professor E. B. VanVleck, of Wesleyan University has received leave of absence and will spend the year abroad.

Professor L. W. Dowling, of the University of Wisconsin, has been granted leave of absence and will spend the coming year mainly in Italy.

Mr. W. H. Roever has been appointed Instructor in Mathematics at the Massachusetts Institute of Technology.

BOOKS AND PERIODICALS.

The Secret of the Circle and the Square. By J. C. Willmon. 12 mo. Cloth, 30 pages. Los Angeles, California: Published by the Author.

This little volume should not be consigned to the junk-pile of literature treating on successful (?) circle-squaring and angle trisecting, as one might infer from its title. The author states in his preface that it is his intention "to demonstrate the possibility of constructing a straight line equal to any given arc of a circle, and through this problem to construct a square equal in area to any circle and a circle equal in area to any square, with solutions of kindred geometrical problems." The constructions are simple, though only approximate, but may, by the method given, be carried out to any desired degree of accuracy. B. F. F.

Plane and Spherical Trigonometry. By P. A. Lambert, Assistant Professor of Mathematics, Lehigh University, and H. A. Foering, Head Master of Bethlehem Preparatory School. 8vo. Cloth, 104 pages. Price, 60 cents. New York: The Macmillan Co.

This text-book gives about the amount of material required of the average college student. The definitions of the functions are given by means of projection, a feature that

is rapidly gaining favor among teachers. The book contains neither tables nor answers to the problems. B. F. F.

Euclid's Parallel Postulate, Its Nature, Validity, and Place in Geometrical Systems. By John William Withers, Ph. D., Principal of the Yeatman High School, St. Louis, Mo. 8vo. Cloth, vii + 192 pages. Chicago: The Open Court Publishing Co.

This is the author's thesis presented to the Philosophical Faculty of Yale University for the Degree of Doctor of Philosophy. The author discusses the subject quite fully in seven chapters. In the first chapter he traces the pre-Lobatchevskian struggle with the parallel postulate; in the second, he considers the discovery and development of non-Euclidean systems; in the third, he notes the general Orientation of the problem; in the fourth, the psychology of the parallel postulate and its kindred conceptions are treated; in the fifth, the nature and validity of the parallel postulate is set forth; and in the sixth, the resulting implications as to the nature of space are noted. The seventh chapter is given a fairly complete Bibliography of non-Euclidean geometry. B. F. F.

The New Knowledge. A popular account of the New Physics and New Chemistry in relation to the New Theory of Matter. By Robert Kennedy Duncan, Professor of Chemistry in Washington and Jefferson College. 8 vo. Cloth, xviii+263 pages. Price, $2.00. New York: A. S. Barnes & Co.

The author of this book has given a very remarkable presentation of the whole field of recent discoveries, ranging from the atoms of the elements, corpuscles, the various rays, radioactivity, and inter-atomic energy, to inorganic evolution and cosmical problems. It is written in a simple though clear style, and, while dealing with the profoundest subjects that ever engaged the attention of man, requires no more of its readers than a love of contemporary natural science and a high school education. The work is one that every person interested in the progress and achievements of the human race, the marvelous inventions and discoveries, will want to read and reread. B. F. F.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. James, Ph. D., Instructor in Mathematics and Astronomy, Washington University, St. Louis. Small 8vo. Cloth, xii+176 pages, 39 figures. Price, $2.00. New York: John Wiley & Sons.

This book is intended for those students who wish to continue the study of Mechanics beyond an elementary course, and is meant to serve as an introduction to advanced treatises. For this reason applications have been almost entirely omitted, to the detriment, it seems to me, of the usefulness of the book, since nothing so firmly clinches a principle as a well selected application of that principle. Problems requiring a knowledge of the calculus and elementary differential equations have been either entirely omitted or approximate solutions only have been given. The book closes with an elementary and concise treatment of Constrained Motions on the Earth's Surface. B. F. F.

Graphic Algebra for Secondary Schools. By H. B. Newson, Ph. D., Associate Professor of Mathematics in the University of Kansas. Pamphlet, 19 pages. Boston and Chicago. Ginn & Co.

This pamphlet is the author's interpretation of the recent Committee of the American Mathematical Association's recommendation both as to kind and amount of graphical work to be used in illustrations and in connection with the solutions of equations. F.

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THE BALLISTIC PROBLEM.

By S. A. COREY.

The problem of the motion of a projectile through a resisting medium, such as air, is discussed at some length by Gilman in the April number of the *Annals of Mathematics*. Of the three methods there given the most accurate is exceedingly laborious. The following method, based on Stirling's formula, is very accurate, simple, and involves but little labor.

It is known that Stirling's formula may be used to develop the following formula:*

$$f(a+x)=f(a)+\frac{x}{m}\left[\frac{f'(a+x)+f'(a)}{2}+f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)\right.$$

$$\left.+f'\left(a+\frac{3x}{m}\right)+\ldots\ldots+f'\left(a+\frac{m-1}{m}x\right)\right]-\frac{B_1\,x^2}{m^2.2\,!}\Big[f''(a+x)-f''(a)\Big]$$

$$+\frac{B_2\,x^4}{m^4.4\,!}\Big[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\Big]-\ldots\ldots$$

$$+(-1)^n\frac{B_n x^{2n}}{m^m.(2n)\,!}\Big[f^{(2n)}(a+x)-f^{(2n)}(a)\Big]+\ldots\ldots(A),$$

B_1, B_2, B_3, etc., being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc.

Using the first two terms only and taking $m=3$, (A) becomes

*See *Annals of Mathematics*, Vol. 5, No. 4, July, 1904.

$$f(a+x)=f(a)+\frac{x}{3}\left[\frac{f'(a+x)+f'(a)}{2}+f'\left(\frac{2x}{3}\right)+f'\left(\frac{x}{3}\right)\right]$$

$$-\frac{x^2}{108}\left[f''(a+x)-f''(a)\right] \text{.........} (1),$$

or, applying (1) to the evaluation of one of the integrals considered by Gilman, viz.,

$$cdx=\frac{dp}{p\sqrt{(1+p^2)}+\log[p+\sqrt{(1+p^2)}]-C}=F'(p)dp \text{........} (2),$$

where $C=3.455425$, $c=.0000339822$, initial angle of elevation $\theta=30^\circ$, $p=\tan\theta=.57735$, and x=distance from muzzle of gun to the point of highest elevation in the flight of the projectile, we have

$$x=\frac{p}{3c}\left[\frac{F'(0)+F'(p)}{2}+F'\left(\frac{p}{3}\right)+F'\left(\frac{2p}{3}\right)-\frac{p^2}{108c}\left[F''(p)-F''(0)\right]\right. \text{........} (3).$$

Differentiating $F'(p)$ with reference to p, and reducing, we get

$$F''(p)=-\frac{2[F'(p)]^2}{\cos\theta}.$$

To permit the use of logarithms to advantage $F'(p)$ may be written

$$1\div\left[\frac{\tan\theta}{\cos\theta}+\log 2+2\log\sin(45^\circ+\tfrac{1}{2}\theta)-\log\cos\theta-C\right] \text{........} (4).$$

Computing $F'(0)$, $F'\left(\frac{p}{3}\right)$, $F'\left(\frac{2p}{3}\right)$, and $F'(p)$ by (4),

$$\begin{aligned}
\tfrac{1}{2}F'(0) &= .14470\\
F'\left(\frac{p}{3}\right) &= .32593\\
F'\left(\frac{2p}{3}\right) &= .37495\\
\tfrac{1}{2}F'(p) &= .22327\\
\text{Sum} &= 1.06885\\
\text{Sum}\div 3 &= .35628
\end{aligned}$$

The first term of (3) then becomes, $\dfrac{.57735\times.35628}{.0000339822}=6053.1$ feet.

The second term of (3) likewise becomes,

$$\frac{.57735^4\left(\frac{.44654^2}{.86603}-.28940^2\right)}{54\times.0000339822}=-26.6 \text{ feet.}$$

Sum of the first and second terms of (3) gives, as the value of x. 6026.5 feet. This result is less than 1 foot in error.

For still greater accuracy either larger values of m may be taken or more than two terms of (A) may be employed.

It is perhaps needless to point out that the same method of evaluating the integrals for y and t may be quite conveniently employed.

OUTLINE OF A COHERENT COURSE IN COLLEGE ALGEBRA.

By DR. A. C. LUNN.

In teaching what may be considered the standard topics in a course on College Algebra it is often difficult to avoid leaving with the students the impression that they have labored over a set of isolated subjects, hard to master because of lack of interrelations. The following outline sketches the result of an attempt to organize many of these subjects into a course, to be developed so that a certain natural unity should be more or less obvious.

§1. The Fundamental Problems.

The main subject is the study of rational functions of a single real variable, in three aspects, the Formula, the Graph, and the Table, denoted in the following by their initial letters. The motto is that every notion, operation, and problem connected with the theory is to be interpreted so far as possible in all three aspects.

By "formula" is understood an algebraic statement of the form $y=f(x)$, giving directions for finding the value of the dependent variable y in terms of the argument x by a certain set of operations, $f(x)$ denoting for the present purpose "an expression in terms of x". The functions or expressions considered here are then either polynomials of the form $A_0x^n+A_1x^{n-1}\ldots\ldots A_{n-1}x+A_n$, or quotients of such, although opportunity is presented for easy passage to certain irrational forms if desired, and certain of the theorems deduced apply to wide classes of functions.

By "graph" is understood a plot of the curve whose equation is $y=f(x)$ in Cartesian coördinates, where the two units of measure are chosen independently according to dictates of convenience for the two axes (not necessarily rectangular), since for present purposes lengths are compared only when parallel.

By a "table" is meant an array of numerical values of y corresponding to assigned values of x, and where not otherwise specified it is understood that the

table is what might be called normal, with the argument values equally spaced; in this case the table includes the columns of differences of various orders as far as found to be needed for completeness.

The primary problems of the theory are then, for a function known in any one of these three aspects, to find means of giving it expression in the other two. This gives six fundamental problems, of which the order of presentation is to some extent arbitrary. The following order is perhaps as good as any.

Problem TG: given the table, to make the graph. This is solved simply by plotting the points given in the table, the inevitable incompleteness of the table being manifest. Practice here is secured well from tables of physical or astronomical observations or statistics, for which no formula is known to the students.

Problem GT: this implies the measurement of chosen ordinates of a given curve, and the insertion of their numerical values in a tabular scheme.

These two problems need be insisted upon only if the idea of plotting is not already somewhat familiar.

Problem FT. For a polynomial this problem is solved for example by the familiar method of synthetic substitution with detached coefficients, implying for each value of the argument an operation such as the following:

$$
\begin{array}{l|llll}
x) & A_0 & A_1 & A_2 \ldots\ldots\ldots\ldots & A_n \\
 & & A_0x & A_0x^2+A_1x & A_0x^n+\ldots\ldots+A_{n-1}x \\
\hline
 & A_0 & A_0x+A_1 & A_0x^2+A_1x+A_2 & A_0x^n+\ldots\ldots+A_n
\end{array}
$$

Many other modes of computation are suggested by the various shapes into which the formulas are thrown for other purposes.

Problem FG. This problem might of course be solved indirectly by computing a table and plotting from the table, or in short, resolving the step FG into the steps FT and TG. But it is much better to do it by purely geometric processes, for which the operations or transformations given in §2 are useful. The manifold ways of using these will be better illustrated in special cases.

Problem TF. Here again the incompleteness of the table is brought vividly home, since it is obviously possible to make many formulas fit the same table of the ordinary kind. But special kinds of solution will be noticed later; in particular the determination of a polynomial of degree n which takes assigned values for $n+1$ given values of the argument.

Problem GF. An outlook is afforded here on the vast fields of mathematical research. If the graph is known to be that of a polynomial of a certain degree the coefficients can be determined by algebraic procees. But the determination of an analytic formula for a given graph is a problem on which has been expended an important part of the mathematical endeavor of a century.

The six problems named, in direct or modified form, make the primary elements of the course, about which all other material is grouped, either as direct outgrowth, or as auxiliary.

§2. The Geometrical Operations.

In the graphic construction of curves which are the graphs of rational functions it is convenient to make use of the following geometric constructions or operations.

(1) The x-Push or Translation through distance a (positive or negative). The curve is moved bodily parallel to the x-axis without changing form, size, or orientation in the plane. The effect on the formula may be indicated thus:

$$ {}^{x}P_{a}f(x)=f(x-a), $$

since the value of the ordinate after the operation must be read as the value previously corresponding to an x-value less by a than its new value.

(2) The y-Push:

$$ {}^{y}P_{b}f(x)=f(x)+b. $$

(3) The x-Stretch or Elongation, with factor m:

$$ {}^{x}S_{m}f(x)=f\left(\frac{x}{m}\right). $$

(4) The y-Stretch, with factor n:

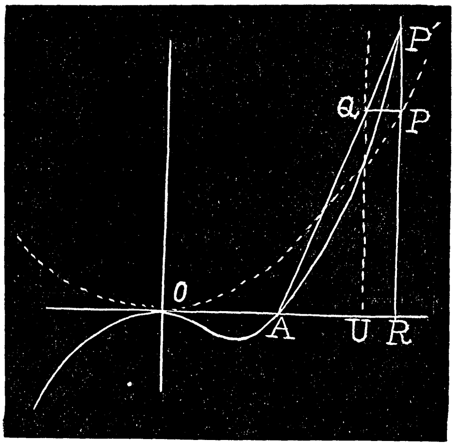
$$ {}^{y}S_{n}f(x)=nf(x). $$

(5) The addition of two curves by algebraic addition of ordinates (operation A).

(6) Multiplication by abscissa difference from assigned point a, where the result is indicated symbolically thus:

$$ M_{a}f(x)=(x-a)f(x). $$

This is accomplished by the following construction,* where the original and the final position of the point are marked P and P'.



A is point of abscissa a; U is point of abscissa $a+1$.

$$ \frac{y'}{y}=\frac{RP'}{RP}=\frac{RP'}{UQ}=\frac{AR}{AU}=x-a, \quad y'=(x-a)y. $$

---

*Used by Professor Moore in course in Calculus since 1902; see also Peano, *Applicazioni del Calçolo Infinitesimale*, p. 74.

Here P is given and P' required; the converse is also needed, P' being the known, P the sought point. The effect of this transformation on a given curve is illustrated in the figure, where the original curve (dotted), is $y=x^2$, the new curve (full line), $y=(x-2)x^2$, with the value $a=2$.

These six operations, which will be denoted by the respective symbols indicated, are to be employed in the construction of a curve when the formula is assigned. Illustrations may now be given.

§3. Standard Expressions and Their Graphs.

Start with the line $y=1$, parallel to the x-axis; then the operation M_0 gives the "standard diagonal," whose equation is $y=x$; another application of M_0 gives the parabola $y=x^2$, and by repetition are obtained all the curves of the form $y=x^n$, where n is a positive integer.

With a new start at the line $y=1$, the inverse operation of M_0, which may be called M_0^{-1}, gives the hyperbola $y=x^{-1}$, and by repetition all curves of the type $y=x^{-n}$. By the use of any reference point A on the x-axis instead of the origin may be obtained any curve of the forms $(x-a)^n$, $(x-a)^{-n}$. Or these may be obtained by the operation ${}^xP_a$, applied to the curves x^n, x^{-n}.

A modification of this set is important. Start with the graph of any expression $f(x)$, which does not meet the axis or go to infinity at the point a, and obtain by direct and inverse operations the curves $(x-a)^nf(x)$, $(x-a)^{-n}f(x)$. It is obvious from inspection of the graphs that the general character of the curves in the neighborhood of the point a is the same as for the symple types $(x-a)^n$, $(x-a)^{-n}$, of corresponding exponent. This is expressed by saying that the point a is a 'root' of order n of the function $(x-a)^nf(x)$, and a 'pole' of order n of the function $(x-a)^{-n}f(x)$.

This prepares the way for the systematic study of polynomials of various degrees and their quotients.

The straight line $y=A_0x+A_1$ is constructed from the diagonal $y=x$ by the successive operations ${}^yS_{A_0}$, ${}^yP_{A_1}$, and the interpretation of A_0 as slope and A_1 as y-intercept thus made clear, to yield a shorter method of construction with ruler.

Or with the equation written in the form $y=A_0(x-a)$, where $a=-A_1/A_0$, the same line may be gotten from $y=x$ by x-push of value a, and y-stretch of value A_0.

Or finally the line $y=A_0$ may be operated upon by M, with reference-point $a=-A_1/A_0$.

As the degree increases, varieties of construction multiply rapidly; no attempt at completeness need be made here.

In the case of the parabolas, or quadratic functions, for instance, there are among others the four important forms:

$$y=A_0x^2+A_1x+A_2,\qquad y=A_0(x^2+px+q),$$
$$y=A_0[(x-a)^2+b],\qquad y=A_0(x-r_1)(x-r_2),$$

the final form being considered only when the roots r_1, r_2 are real.

The first form is constructed from the line $y = A_0 x + A_1$, by the successive operations M_0, which yields $A_0 x^2 + A_1 x$, and ${}^yP_{A_2}$ which gives the required form. This is plainly the graphic analogue of the synthetic substitution in the formation of the table by computation.

The second form is obtained from the line $y = x + p$ by the operations M_0, ${}^yP_q$, ${}^yS_{A_0}$.

The third form comes from the standard parabola $y = x^2$, by means of ${}^xP_a$, which yields $(x-a)^2$, then ${}^yP_b$ which gives $(x-a)^2 + b$, then finally ${}^yS_{A_0}$.

The fourth form results from application of M_{r_2} to the line $A_0(x - r_1)$, or of M_{r_1} to the line $A_0(x - r_2)$.

The character of b as negative, 0, or positive, gives the classification of quadratic expressions as having two, one, or no real roots.

There are here four different choices of the set of three constants needed to determine the parabola, sets (A_0, A_1, A_2), (A_0, p, q), (A_0, a, b), (A_0, r_1, r_2). The determination of the connections between these sets affords a fairly extensive exercise in the manipulation of formulas, the complete problem being to express in terms of the constants of any set those of the remaining three.

For the further illustration of the methods of construction for higher degrees, it will perhaps be sufficient to consider a definite case in some detail.

Let the formula be $y = 2x^3 - 4x^2 - 22x + 24$. Alternative constructions are then:

(I) Operate on line $2x - 4$, by M_0, giving $2x^2 - 4x$; then by ${}^yP_{-22}$, giving $2x^2 - 4x - 22$; then by M_0, giving $2x^3 - 4x^2 - 22$; then by ${}^yP_{-22}$, (synthetic substitution).

(II) Write the formula $y = 2[(x - \frac{2}{3})^3 - \frac{29}{3}x + \frac{332}{27}]$. Here give the standard cubic x^3 an x-push of value $\frac{2}{3}$, add the ordinates of the line $-\frac{29}{3} + \frac{332}{27}$, and make y-stretch with factor 2.

(III) The cubic can be factored to $2(x-4)(x-1)(x+3)$. Operate on line $2(x-4)$ by M_1, then by M_{-3}.

The table of the same cubic may have the form:

x	$f(x)$	$\triangle_1$	$\triangle_2$	$\triangle_3$
3.0	-24.00			
		$+7.75$		
3.5	-16.00		8.50	
		$+16.25$		1.50
4.0	0.00		10.00	
		$+26.25$		1.50
4.5	$+26.25$		11.50	
		$+37.75$		1.50
5.0	$+64.00$		13.00	
		$+50.75$		
5.5	114.75			

§4. Algebraic Theorems.

Sufficient experience in construction of graphs belonging to a variety of formulas leads by induction to the formulation of a number of theorems of which a few may be written down here.

If the graph of a rational function meets the x-axis at the point a, after the manner of the elementary function $(x-a)^n$, its algebraic expression contains the factor $(x-a)^n$ in the numerator; if it goes to infinity like the function $(x-a)^{-n}$ it has the factor $(x-a)^{-n}$ in the denominator. These considerations lead to the general factor-theorem, with the notions of simple and multiple roots and asymptotes, and the attempt to represent the given function by *addition* of elementary functions, each with a single asymptote, developes into a study of resolution into partial fractions.

If the slopes of the secant-lines are obtained by division of the first differences in the table by the x-difference, and their values plotted into a set of curves, one for each chosen value of the x-difference, these curves, "secant-slope curves," are seen to approach coincidence with the tangent-slope curve, or "derived curved," which gives the slope of the tangent line as a function of x. Similarly the rth differences divided by the rth power of the x-difference give curves which approach the rth slope-curve, yielding the successive derivatives of the function. This mode of attack is however, in the case of the higher derivatives, convenient mainly for the polynomials.

Inspection of tables computed from polynomials of various degrees suggests that for a polynomial of degree n the first differences act like a function of degree $n-1$, the second like one of degree $n-2$, and so on, the nth differences being all alike, or a function of degree 0. The proof of this is obtained through the agency of the binomial theorem, proved from the theory of combinations or by induction, permitting the expansion of $(x+w)^n$, $(x+2w)^n$, ..., where w is the value of the x-space of the table, and there may be obtained here directly the formulas for the successive derivatives of x^n, together with the theorems that multiplication by a constant multiplies the derivative, and that the derivative of a polynomial is the sum of the derivatives of the separate terms. Then finally the expansion of any polynomial of degree n is seen to take the form of Taylor's theorem for the expansion of $f(x+h)$, but ending with the nth power of h because of the vanishing of the higher derivatives.

The binomial coefficients ("Pascal Triangle") appear also if an error is made in some entry of the table, this error being propogated into the columns of differences, magnified according to the binomial coefficients with alternating signs.

§5. Interpolation Formula. Linear Equations. Determinants.

Problem TF may be solved in a special case in the form: given $n+1$ entries in a table to find a polynomial of degree n yielding those values. One method of solution is to consider the required function as the sum of n polynomials, each of which goes through one of the given points and the projections

of the remaining n points on the x-axis. This yields immediately the classic interpolation formula of LaGrange.

Another solution is to determine the coefficients directly from the linear equations defining the conditions imposed. This introduces the theory of determinants, which is also suggested by the problem of indeterminate coefficients in the partial fraction expansions, and by the problem of finding the intersection of two linear graphs.

§6. Converse Problem. Determination of Roots.

The previous discussion has considered chiefly the problem, given the value of the argument x, to find the value of the rational function $f(x)$. The converse of this, given the value of $f(x)=b$, to find the value or values of x, is immediately seen to be equivalent to the problem, find the roots of the rational function $f(x)-b$, *i. e.*, the roots of the numerator when the function is written as the quotient of two polynomials. The numerical work is carried out by the Horner method of well-directed trial and error, with change of origin and synthetic substitution.

The preceding sketch is plainly rough and incomplete, but is perhaps sufficient to indicate a trend of thought which has been found to yield abundant material for a quarter's work, without sacrifice of unity of structure. Many other topics may be connected easily with those indicated if time permits.

GRAPHICAL METHODS IN TRIGONOMETRY.

By DR. L. E. DICKSON.

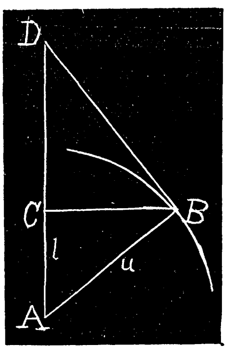
Aside from the important work on the solution of triangles by diagrams drawn to scale, graphic methods are not usually employed in trigonometry. Even if the cartesian graphs of the trigonometric functions are constructed, no serious applications are made of these graphs. They are, however, admirably adapted to the explanation of interpolation, to the visualization and retention in the memory of the ratios for the angles 0°, 90°, etc. (in contrast to their derivation as limiting values), and to the natural solution of trigonometric equations,—in particular, to the visualization of the number of angles $<180°$ having a given sine or cosine. In addition to these minor advantages resulting from a frequent appeal to the graphs, the graphic method may be employed to perform the highly important service of leading the student naturally to the majority of the fundamental trigonometric formulae, including the addition theorem and formulae for conversion of sum into product. This is in marked contrast to the current method by which each formula makes its appearance from some unseen source, to be followed by a more or less artificial proof.

An inspection of the cartesian sine curve reveals two facts (proved by recurring to the unit circle and ordinates used in constructing the graph): the symmetry of each arch and the equality of the various arches. Hence if the sine wave is moved to the right 180° and then reflected on the scale line, it coincides with its former trace as a whole;* hence $\sin(180°+\alpha)=-\sin\alpha$, for *every* angle α. Rotation of the curve about 0° through an angle 180° leads similarly to $\sin(-\alpha)=-\sin\alpha$, for every α. Reflection of the curve on the vertical through the point marked 90° leads to $\sin(180°-\alpha)=\sin\alpha$, for every α. Moving the sine wave 90° to the left, we obtain the cosine wave; hence $\sin(90°+\alpha)=\cos\alpha$, for every α. Performing the last two operations, we get $\sin\alpha=\sin(180°-\alpha)=\cos(90°-\alpha)$. Similarly, all the formulae of this type follow immediately from a combination of three of the preceding operations.

A valuable exercise is afforded by the composition of waves of different periods and phases, with emphasis on the physical applications.

As it seems preferable to *define* the cosecant, secant, and cotangent as the reciprocals of the sine, cosine, and tangent, respectively, it is desirable to construct the graphs of the former direct from the latter.

The following construction for the reciprocal $AD=r$ of a given directed line $AC=l$, the unit of length being u, is very convenient, since it yields r in the position desired for the reciprocal graph. We determine B as an intersection of a circle of center A and radius u with the perpendicular to AC at C. We make angle $ABD=90°$. Then $l:u=u:AD$.



Cartesian graphs may be converted mechanically into polar coördinate graphs. Take a rectangle $ABCD$, whose length AB is approximately π times its height BC, and make alternate forward and backward narrowfolds parallel to BC. The rectangle is compressed into a fluted surface, with D near C, and A near B. If we hold A and B together, but allow the end DC to open, we ultimately obtain a *fan*, whose outline is approximately a semi-circle.† If on the original rectangular strip appeared a cartesian sine arch with ends at A and B (the unit not being too large) and a part of the U-shaped cosecant graph, there will appear on the fan a circle and tangent straight line, representing the polar graphs of sine and cosecant.

*Hence the equation or graph $y=\sin x$ is transformed into itself by

$$T \;:\; x'=x+180°,\; y'=-y;$$

likewise by the rotation through angle 180° about 0°, viz.,

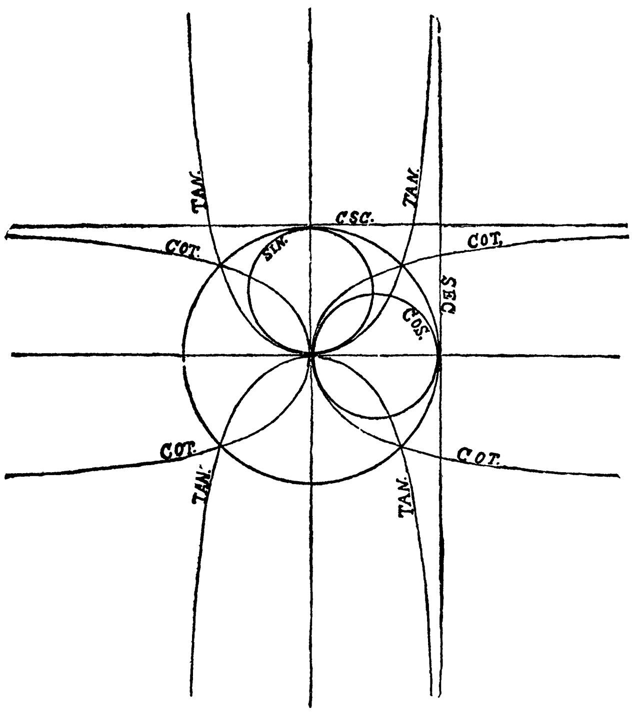
$$S \;:\; x'=-x,\; y'=-y.$$

The product $R=ST$ gives the reflection on the 90° vertical:

$$R \;:\; x'=180°-x,\; y'=y.$$

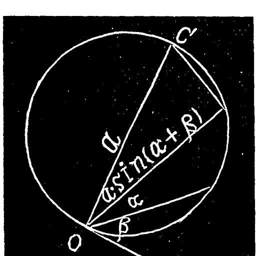
The infinite group transforming $y=\sin x$ into itself is generated by T and S.

†Let the cartesian coordinates of a point P of the rectangle be x, y, the x-axis being AB, the y-axis a perpendicular to AB at its center O. On the fan let the polar coordinates (origin O) of P be r, A. Then $r=y$, while the complement of A is measured by an arc of length x on the semi-circle.



This mechanical derivation of the polar graphs should be followed by their geometric construction on polar coördinate paper, and later by the simple formal proofs that the polar graphs of the sine and cosine are circles with respectively horizontal and vertical tangents at the origin O, and that the graphs of the cosecant and secant are straight lines tangent at the opposite end of the diameter through O. More generally, the polar graph of $r = a\sin(\alpha+\beta)$, where β is a constant angle, is the circle of diameter a, whose tangent at O makes the angle β with the initial line. The graphs of $r = a\sin\alpha$ and $r = a\cos\alpha$ are obtained by setting $\beta = 0°$ and $\beta = 90°$, respectively.

While the composition of waves on cartesian paper has important physical applications, the composition of the polar graphs offers greater interest as well as greater importance in the mathematical theory. We may give the following definition of composition of graphs: The points (r_1, α), (r_2, α) on the two polar graphs lead to the point $(r_1 + r_2, \alpha)$ on the *compound* graph. The following theorem is fundamental:



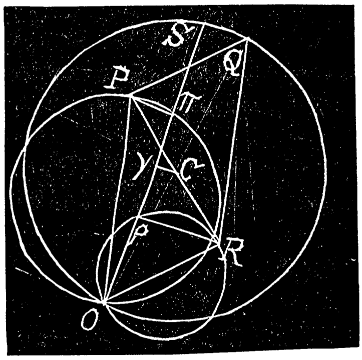
The compound of a circle on the diameter OP with a circle on the diameter OR is the circle having as diameters the diagonal OQ of the parallelogram OPQR.

Let ρ and π be the intersections of an arbitrary line through O with the given circles, and take $\pi S = O\rho$. We are to prove that S lies on the circle with center C at the middle point of PR and radius OC. Draw the perpendicular $C\gamma$ from C to OS. It is to be shown that $O\gamma = S\gamma$. This follows from $\rho\gamma = \pi\gamma$, a consequence of equal intercepts PC and CR between the parallels $P\pi$, $C\gamma$, $R\rho$.

Consider the special case in which $POR = 90°$. Set $a = OP$, $b = OR$. Then the equations of the circles on OP, OR, OQ are respectively,

$$r = a\sin\alpha, \quad r = b\cos\alpha, \quad r = \sqrt{(a^2 + b^2)}\sin(\alpha+\beta),$$

where $\beta = POQ$=angle between OR and the tangent at O to the circle on OQ. Hence



$$a\sin\alpha + b\cos\alpha = \sqrt{(a^2+b^2)}\sin(\alpha+\beta)$$

for every angle α. But

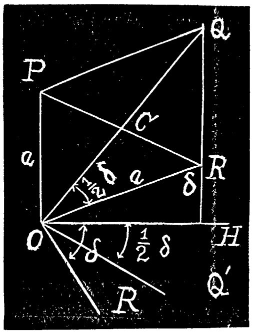
$$a = \sqrt{(a^2+b^2)}\cos\beta,\ b = \sqrt{(a^2+b^2)}\sin\beta.$$

Hence

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Consider next the special case $OP = OR = a$. Let OR' and OQ' be the tangents at O to the circles on the diameters OR and OQ. The initial line OH is perpendicular to OP. The circles on OP and OR are $r = a\sin\alpha$, $r = a\sin(\alpha+\delta)$, where $\delta = HOR'$. Since ORQ is isosceles, the exterior angle $\delta = 2ROQ$. Hence $\frac{1}{2}\delta = ROQ = HOQ'$ (the arms of the angles being perpendicular). Hence the compound circle on the diameter OQ is

$$r = (2a\cos\tfrac{1}{2}\delta)\sin(\alpha+\tfrac{1}{2}\delta).$$



Hence

$$\sin\alpha + \sin(\alpha+\delta) = 2\cos\tfrac{1}{2}\delta\,\sin(\alpha+\tfrac{1}{2}\delta).$$

Since α and δ are arbitrary angles, this formula includes the four formulae expressing $\sin\alpha \pm \sin\gamma$ and $\cos\alpha \pm \cos\gamma$ as products.

Further interesting exercises are obtained from the graphs of $r = b\sin(k\alpha+\delta)$ and their composition with $r = a\sin\alpha$.

As it is desirable to have two essentially different proofs of each theorem, the following alternative proof* of the addition theorem is suggested on account of its simplicity. Let the sum of the given angles α and β be less than 180° (the modifications of the picture for other cases being obvious). It is assumed that the sine of proportion has been previously proved.

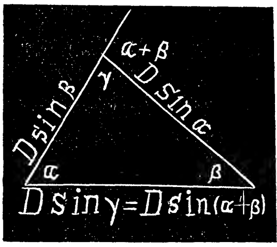
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D \text{ (=diameter of circumscribed circle).}$$

Since the projection of one line on another equals the product of its length by the cosine of the included angle, we obtain from the figure,

$$D\sin(\alpha+\beta) = D\sin\alpha\cos\beta + D\sin\beta\cos\alpha.$$

*Compare Veblen's proof, Monthly, January, 1904, p. 7.

Likewise it seems desirable to supplement the usual analytic derivation of



$\tan\frac{1}{2}A = r/(s-a)$ by a purely geometric proof (MONTHLY, 1902, p. 36).

Theorem. *The compound of a sphere on the diameter OP with a sphere on the diameter OR is a sphere having as diameter the diagonal OQ of the parallelogram OPQR.*

The proof is similar to the above for circles. If three parallel planes make equal intercepts on one transversal they make equal intercepts on any other transversal.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

228. **Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.**

Sum the infinite series

$$\frac{1}{11.13}+\frac{1}{23.25}+\frac{1}{35.37}+\frac{1}{47.49}+\frac{1}{59.61}+\cdots\cdots \quad \text{[Oxford, 1895].}$$

Solution by the PROPOSER.

We can show that*

$$\frac{1}{2\theta}\left[\frac{1}{\theta}-\cot\theta\right]=\frac{1}{(\pi-\theta)(\pi+\theta)}+\frac{1}{(2\pi-\theta)(2\pi+\theta)}+\frac{1}{(3\pi-\theta)(3\pi+\theta)}+\cdots\cdots$$

Put $\theta=\pi/12$ and we get

$$\frac{\pi}{6}\left[\frac{12-\pi\cot 15^\circ}{\pi}\right]=\frac{144}{\pi^2}\left[\frac{1}{11.13}+\frac{1}{23.25}+\cdots\cdots\right].$$

Hence the required sum $=\dfrac{12-\pi\cot 15^\circ}{24}$.

Also solved by J. Scheffer.

229. **Proposed by B. F. YANNEY, Mount Union College, Alliance, O.**

If $a_1{}^n+a_2{}^n+a_3{}^n+\cdots\cdots+a_r{}^n=A^n$, $a_1{}^m+a_2{}^m+a_3{}^m+\cdots\cdots+a_r{}^m>$ or $<A^m$, according as $m<$ or $>n$; provided all the letters stand for positive real numbers.

No satisfactory solution has been received.

*Expand $\sin\theta$ in factors, take logarithms of each expression, and differentiate.

230. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Find the value of the determinant of n rows,

$$\begin{vmatrix} 5 & 2 & 0 & 0 & 0 & \ldots\ldots \\ 2 & 5 & 2 & 0 & 0 & \ldots\ldots \\ 0 & 2 & 5 & 2 & 0 & \ldots\ldots \\ 0 & 0 & 2 & 5 & 2 & \ldots\ldots \\ 0 & 0 & 0 & 2 & 5 & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}$$

[Oxford, 1900.]

Solution by the PROPOSER.

Denote the determinant by u_n. Then $u_n=5u_{n-1}-4u_{n-2}$.
Let $S=u_1+u_2x^2+u_3x^2+\ldots\ldots$ Then

$$(1-5x+4x^2)S=u_1+(u_2-5u_1)x=5-4x.$$

$$S=\frac{5-4x}{1-5x+4x^2}=\tfrac{1}{3}\left[\frac{16}{1-4x}-\frac{1}{1-x}\right].$$

Then $u_n=\text{co}\, x^{n-1}$ in $S=\frac{1}{3}[4^{n+1}-1]$.

Also solved by M. E. Graber.

231. Proposed by O. L. CALLECOT, Omaha, Neb.

Sum to infinity: $\dfrac{1}{2.3.4}+\dfrac{1}{5.6.7}+\dfrac{1}{8.9.10}+\ldots\ldots$

Remark by Editor EPSTEEN.

Mr. A. H. Holmes notes that

$$\frac{2}{(2+x)(2+x+1)(2+x+2)}=\frac{1}{2+x}-\frac{2}{2+x+1}+\frac{1}{2+x+2}$$

and then substitutes $x=0$, 1, 2, ... From this point on it seems to me that his solution should proceed thus:

$$\frac{1}{1.2.3}+\frac{1}{5.6.7}+\frac{1}{8.9.10}+\ldots\ldots$$

$$\begin{aligned}=\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}&\\ +\tfrac{1}{3}-\tfrac{2}{4}+\tfrac{1}{5}&\\ +\tfrac{1}{4}-\tfrac{2}{5}+\tfrac{1}{6}&\\ +\tfrac{1}{5}-\tfrac{2}{6}+\tfrac{1}{7}&\\ \vdots\qquad\vdots&\end{aligned}$$

$=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.

232. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If one person out of 50 die annually and one person out of 30 is born annually, how long at this rate would be required for the population to treble itself?

Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

Let a be the population at the beginning; then at the end of the first year the population will be $a+(\frac{1}{30}-\frac{1}{50})a=\frac{76}{75}a$; at the end of the second year $\frac{76}{75}$ of $\frac{76}{75}a=(\frac{76}{75})^2a$; and at the end of the nth year $(\frac{76}{75})^na$.

Hence $(\frac{76}{75})^na=3a$, whence $n=\dfrac{\log 3}{\log 76-\log 75}=82.94$, or nearly 83 years.

Also solved by J. Scheffer, M. E. Graber, F. R. Honey, R. D. Carmichael, A. H. Holmes, G. W. Greenwood, L. E. Newcomb, and the Proposer.

233. Proposed by J. T. KEYES, Fogg High School, Nashville, Tenn.

At what time between 10 and 11 o'clock is the second hand of a clock one minute space nearer to the hour hand than it is to the minute hand?

I. Solution by J. SCHEFFER, Hagerstown, Md.

Let the time be x minutes past 10 o'clock. We assume that at the beginning of every minute the second hand points at 12 on the dial. The distance of the second hand at the required time is $60x-x=59x$; and that of the second hand from the hour hand is $60-60x-(10-\frac{1}{12}x)=50-60x+\frac{1}{12}x$.

$\therefore 59x=51-60x+\frac{1}{12}x$, whence $x=\frac{612}{1427}$ minutes$=25\frac{1045}{1427}$ seconds.

II. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

Let us measure the spaces clockwise from the minute hand to the second hand, and from the second hand to the hour hand. At n minutes and t seconds after ten, the number of minute spaces after the hour at which the hour hand, minute hand, and second hand stand are, respectively,

$$50+\frac{n}{12}+\frac{t}{720},\quad n+\frac{t}{60},\quad t.$$

$$\therefore 50+\frac{n}{12}+\frac{t}{720}-t+1=t-\left(n+\frac{t}{60}\right);\ i.\ e.,\ 36720+780n=1427t.$$

By putting $n=1, 2, \ldots\ldots, 59$, we get the corresponding values of t.

Also solved by Frederic R. Honey, A. H. Holmes, and L. E. Newcomb.

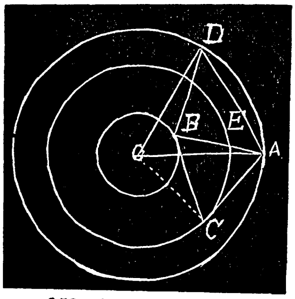
GEOMETRY.

257. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Construire un triangle équilatéral sachant qu'el doit s'appuyer par ses trois sommets sur trois circonférences concentriques données. *Rouché et Comberousse.*

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Draw any radius OA of the largest circle, and on this construct the equilateral triangle OAD. From D as center with a radius OE of the middle circle draw an arc cutting the smallest circle at B, so that $DB=OE$, and draw BA, make $AC=AB$, and connect C with B, then ABC be the required triangle.



For, in the two triangles ABD and ACO, $AD=AO$, $BD=OC$, $AB=AC$.

$\therefore \triangle ABD = \triangle ACO$; $\therefore \angle DAB = \angle OAC$.

$\therefore \angle DAO = \angle BAC = 60^\circ$.

$\therefore \triangle ABC$ is an equilateral triangle having its vertices in the three concentric circumferences.

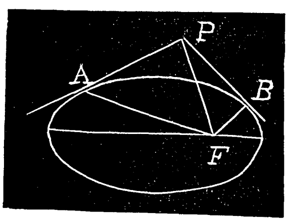
G. W. Greenwood called attention to the similarity of this problem to No. 250.

258. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Mo.

Prove that the tangents to an ellipse from any external point subtend equal angles at the focus, by means of the formula $\tan\phi = (m_1 - m_2)/(1+m_1 m_2)$, where ϕ is the angle between the focal radius of either of the points of tangency and the line joining the focus and the external point, and m_1 and m_2 are the slopes of these two lines.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Designate the coördinates of the point of contact A by x_1, y_1, and those of the point of contact B, x_2, y_2, the equation of the ellipse being $a^2y^2 + b^2x^2 = a^2b^2$; then the equation of tangents PA and PB will be, respectively, $a^2yy_1 + b^2xx_1 = a^2b^2$, and $a^2yy_2 + b^2xx_2 = a^2b^2$. By solving these as two simultaneous equations we find



$$x = \frac{a^2(y_2 - y_1)}{x_1y_2 - x_2y_1},\quad y = -\frac{b^2(x_2 - x_1)}{x_1y_2 - x_2y_1},$$

which are the coördinates of the point P. Since the coördinates of focus F are $-ae$ and 0, we find the slope of PF

$$= -\frac{\dfrac{b^2(x_2 - x_1)}{x_1y_2 - x_2y_2}}{\dfrac{a^2(y_2 - y_1)}{x_1y_2 - x_2y_1} + ae} = -\frac{b^2(x_2 - x_1)}{a[ay_2 - ay_1 + e(x_1y_2 - x_2y_1)]},$$

and the slope of $AF = \dfrac{y_1}{x_1 + ae}$.

$$\therefore \tan PFA = \left[-\frac{b^2(x_2 - x_1)}{a[ay_2 - ay_1 + e(x_1y_2 - x_2y_1)]} + \frac{y_1}{x_1 + ae}\right]$$

$$\div\left[1 - \frac{b^2y_1(x_2 - x_1)}{a(x_1 + ae)[ay_2 - ay_1 + e(x_1y_2 - x_2y_1)]}\right]$$

$$=-\frac{(a+ex_1)[a^2y_1y_1+b^2x_1x_2-a^2b^2)]}{a^2(a+ex_1)[x_1y_2-x_2y_1-ae(y_1-y_2)]}=-\frac{a^2y_1y_2+b^2x_1x_2-a^2b^2}{a^2[x_1y_2-x_2y_1-ae(y_1-y_2)]}.$$

It is seen that this expression is symmetrical with reference to x_1 and x_2, y_1 and y_2 with the exception of the sign, but considering that by finding tanPFB the slope of BF comes first, it is at once seen that tanPFB is the same as tanPFA. The difficulty of this method lies in the complicated algebraic work, which is avoided by using polar coördinates.

Solution of 255 by Prof. William Hoover was received after the solution in last issue had gone to press. Also a solution of 256 was received from a contributor who failed to sign his name.

NOTE. Professor Matz sent in a solution of 254 in which he points out that the line $x-4a=0$ is both tangent and normal to the curve. But the solution is not general. Who can give a general solution?

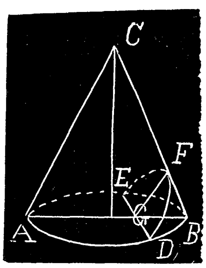
CALCULUS.

195. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, O.

Given a right cone of altitude h and radius r, to locate the plane parallel to its side which bisects the cone.

Solution by A. H. HOLMES, Brunswick, Maine, and J. SCHEFFER, Hagerstown, Md.

Let, in the right cone CAB, DEF represent a parabolic section. Put $BG=x$, $GE=y$, $FG=z$. The area of $DEF=\frac{4}{3}yz$; and consequently the volume of



$$BDEF=\tfrac{4}{3}\cdot\frac{h}{\sqrt{(r^2+h^2)}}\int_0^a yzdx,$$

and since $y^2=2rx-x^2$, and $z=\frac{x}{2r}\sqrt{(r^2+h^2)}$, we have for the volume, the integral

$$\tfrac{2}{3}\frac{h}{r}\int_0^x xdx\sqrt{(2rx-x^2)}=\tfrac{2}{3}\frac{h}{r}\left[\tfrac{1}{2}r^3\cos^{-1}\frac{r-x}{r}-\frac{3r^2+rx-2x^2}{6}\sqrt{(2rx-x^2)}\right].$$

To determine x for the condition that this volume is to be half the cone, we have the transcendental equation

$$2r^3\cos^{-1}\frac{r-x}{r}-\tfrac{2}{3}(3r^2+rx-2x^2)\sqrt{(2rx-x^2)}=r^3.$$

An approximate value of x is $x=1.3r$.

Also solved by R. D. Carmichael.

196. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

The shortest tangent intercepted by the axes of the ellipse to which the tangent is drawn, equals the sum of the semi-axes of the ellipse.

I. Solution by the PROPOSER.

Tangent$=y\surd[1+(dx/dy)^2]+x\surd[1+(dy/dx)^2]$, in which $y=(b/a)\times\surd(a^2-x^2)$ and $dy/dx=-bx/a\surd(a^2-x^2)$.

$$\therefore U=\frac{1}{a}\left[\frac{\surd(a^2-x^2)}{x}+\frac{x}{\surd(a^2-x^2)}\right]\surd[a^4-(a^2-b^2)x^2]$$

$$=a\sqrt{\left(\frac{a^4-(a^2-b^2)x^2}{x^2(a^2-x^2)}\right)}=\text{a minimum.}$$

$\therefore x=a^3/(a+b)$; and, consequently, the length of the required tangent becomes as stated in the problem.

II. Solution by G. W. GREENWOOD, M.A., Professor of Mathematics, McKendree College, Lebanon, Ill., and J. SCHEFFER, Hagerstown, Md.

Denote the length of the tangent by l, and its equation by $y=mx+\surd(a^2m^2+b^2)$.

$$\therefore l^2=(1+\frac{1}{m^2})(a^2m^2+b^2).$$

$$l^2m^2=a^2m^4+b^2+a^2m^2+b^2m^2=(am^2-b)^2+m^2(a+b)^2.$$

$$\therefore l^2=(a+b)^2+\left(\frac{am^2-b}{m}\right)^2.$$

Hence the minimum value of l is $a+b$.

Also solved by M. E. Graber, and W. L. Tryon.

197. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

$$\int_0^\infty \frac{\sin mx \cos nx}{x}dx.$$

Solution by G. W. GREENWOOD, M. A., Lebanon, Ill.; M. E. GRABER, Tiffin, Ohio, and the PROPOSER.

The required integral may be written

$$\tfrac{1}{2}\int_0^\infty \left[\frac{\sin(m+n)x}{x}+\frac{\sin(m-n)x}{x}\right]dx,$$

and it therefore reduces to problem No. 186, [January, 1905, page 22]. If $m+n$ and $m-n$ are both positive, the result is $\frac{1}{2}\pi$. If both negative, $-\frac{1}{2}\pi$. If of opposite sign, 0.

Also solved by S. A. Corey.

189. Proposed by M. E. GRABER, A. M., Heidelburg University, Tiffin, O.

Show that $e\int_1^\infty$, $e^2\int_2^\infty$, ...$e^n\int_n^\infty$, ... are integers divisible by $(p+1)!$, when the expression under the integral is $z^p\Big[(z-1)\ldots\ldots(z-n)\Big]^{p+1}e^{-z}\,dz$.

Solution by the PROPOSER.

$\int_0^\infty z^p[(z-1)\ldots\ldots(z-n)]^{p+1}e^{-z}dz$ is divisible by $\rho\,!$ owing to the well known relation $\int_0^\infty z^p e^{-z}dz=\rho\,!$. Likewise, by substituting $z=z'+1$, $z=z'+2$, ..., $z=z'+n$, it can be seen that $e\int_1^\infty$, $e^2\int_2^\infty\ldots\ldots e\int_n^\infty\ldots\ldots$ are divisible by $(\rho+1)\,!$.

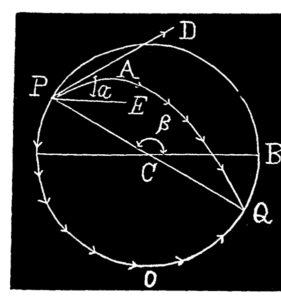
MECHANICS.

173. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A squirrel is in a cylindrical cage and oscillates with it about its axis which is horizontal. At the instant when he is at the highest point of the oscillation, he leaps to the opposite extremity of the diameter and arrives there at the same instant as the point at which he left. Determine his leap completely.

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let P be the highest point to which the squirrel ascends in his oscillations, the coördinates of this point being x_0, y_0; and Q the point opposite to which he jumps, the coördinates of this point being x_1, y_1, the origin of coördinates being taken at O.



Let the angle $BCP=\beta$, and let α be the angle between the horizon and the line of direction taken by the squirrel to reach the point Q.

Since the squirrel and cage are at rest when the squirrel reaches the highest point of his oscillations, and since, at that instant, the squirrel jumps to the point, Q, diametrically opposite, the cage is set in uniform motion in the direction POQ by the impulsive force of the squirrel's jump.

Let F_s be the impulsive force of the squirrel's jump; F_c, the effective impulsive force received by the cage; M_s, the mass of the squirrel; M_c, the mass of of the cage; V_s, the velocity of the squirrel; and V_c, the velocity of the cage. Then

$$F_c=F_s\sin(\beta-\alpha)\ \ldots\ldots\ldots(1),\quad F_c\tau=M_cV_c\ \ldots\ldots\ (2),\ \text{and}\ F_s\tau=M_sV_s\ \ldots\ldots\ (3).$$

The time for the point, P, to have the position, Q, is $t=\pi R/V_c$, and the time for the squirrel to reach the same point, by the parabolic path PAQ is $t=-\dfrac{2R\cos\beta}{\cos\alpha}$. Hence, $-\dfrac{2R\cos\beta}{V_s\cos\alpha}=\dfrac{\pi R}{V_c}\ \ldots\ldots\ (4)$.

From (1), $F_c/F_s=\sin(\beta-\alpha)$, and from (2) and (3), $F_c/F_s=M_cV_c/M_sV_s$. Hence, $M_cV_c/M_sV_s=\sin(\beta-\alpha)$. Hence, $V_c=M_sV_s\sin(\beta-\theta)/M_c$.

Substituting this value of V_c in (4), and solving for α, we have

$$\tan\alpha=\tan\beta+\frac{\pi M_c}{2M_s\cos^2\beta}.$$

Knowing α, the other quantities may be found.

176. Proposed by A. H. HOLMES, Brunswick, Me.

A solid cube weighs 300 pounds. If a power is applied at an angle of 45° at an upper edge of the cube, how many foot-pounds will be required to overturn the cube?

Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

In order to overturn the cube it must be revolved on a lower edge until the center of mass is vertically over that edge, and this will require the lifting of the 300 pounds through a distance $a(\sqrt{2}-1)$, a being the edge of the cube, against gravity.

$\therefore$ the work done$=300a(\sqrt{2}-1)=124.26a$ foot-pounds. Hence, the *size* of the cube can not be left out of the calculation.

178. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

A weight W is drawn up a rough conical hill of height h and slope α. and the path cuts all the lines of greatest slope at the constant angle ϕ. Find the work done in attaining the summit.

[Problem 11, page 226, *Johnson's Theoretical Mechanics.*]

Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The weight W can be resolved in forces $W\cos\alpha$ perpendicular to the surface, $W\sin\alpha\cos\phi$ along its path, and $W\sin\alpha\sin\phi$ perpendicular to these directions. The force required to move the body is therefore $\mu W\cos\alpha+W\sin\alpha\cos\phi$.

The path is the equiangular spiral $s=r\sec\phi$, and hence its length from the foot of the hill is $n\operatorname{cosec}\alpha\sec\phi$.

The work done is then

$$n\operatorname{cosec}\alpha\sec\phi(\mu W\cos\alpha+W\sin\alpha\cos\phi);\ i.\ e.,\ Wh(\mu\cot\alpha\sec\phi+1).$$

179. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the *velocity* of a body moving under an acceleration tending to the center *varies* as the radius of curvature, the body will describe a cycloid.

Solution by the PROPOSER.

Assume $x=\text{ver-sin}^{-1}(y)-\sqrt{(2ry-y^2)}$ to represent the orbit; then $R,=v,=2\sqrt{(2ry)}$, which fulfills the conditions of the problem.

DIOPHANTINE ANALYSIS.

126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

Solution by A. H. HOLMES, Brunswick, Maine.

The order of play would be as follows, the first eight letters representing the players:

$$\left.\begin{array}{l} A-B \text{ vs. } C-D \text{ and } E-F \text{ vs. } G-H \\ A-B \text{ vs. } E-F \text{ and } C-D \text{ vs. } G-H \\ A-B \text{ vs. } G-H \text{ and } C-D \text{ vs. } E-F \end{array}\right\} \text{first evening.}$$

The arrangement of couples for the second evening would be: A–C vs. B–D and E–G vs. F–H, alternating as first evening.

Third evening: A–D vs. F–G and B–C vs. E–H.

Fourth evening: A–E vs. B–F and C–G vs. D–H.

Fifth evening: A–F vs. B–G and C–H vs. D–E.

Sixth evening: A–G vs. B–H and C–E vs. D–F.

Seventh evening: A–H vs. C–F and B–E vs. D–G.

It would therefore take seven evenings to *make* the series, or six evenings to *complete* the series.

AVERAGE AND PROBABILITY.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

A box contains n tickets numbered from 1 to n. How many draws, on the average, will it take to draw all the numbers, each ticket being replaced before drawing again? What is the numerical result for $n=2$ and $n=6$?

REMARKS. Mr. Corey and the Proposer insist that the published solution of this problem in the May Number is incorrect. Mr. Sanders gets for $n=2$, $p=\frac{1.2}{2}+\frac{1.3}{2^2}+\frac{1.4}{2^3}+\frac{1.5}{2^4}+\ldots\ldots=3+$; and for $n=3$, $p=\frac{2.3}{3^2}+\frac{2.4}{3^2}+\frac{14.5}{3^4}+\frac{10.6}{3^4}+\frac{62.7}{3^8}+\ldots\ldots$, the sum of the first ten terms of which is 5.1. From a large number of actual trials, he obtained the following results: $n=2$, $p=2.9$; $n=3$, $p=5.5$; $n=4$, $p=8.5$; $n=5$, $p=12.31$; $n=6$, $p=15.67$; $n=8$, $p=22.66$; $n=12$, $p=41$,—values which are (he observes) approximately equal to $\sqrt{n^3}$.

As stated in the May Number, the published solution conforms to the solution of Problem IX, page 52, of Meyer's *Wahrscheinlichkeitsrechnung*, which problem I take to be similar to the one under consideration. The formula used by Professor Landis in his solution is the same as that obtained by Meyer, a formula which is only approximate, but holds with great exactness if n is large. The correct result is to be obtained from the equation

$$\tfrac{1}{2}=1-n\left(\frac{n-1}{n}\right)^i+\frac{n(n-1)}{2!}\left(\frac{n-2}{n}\right)^i-\frac{n(n-1)(n-2)}{3!}\left(\frac{n-3}{n}\right)^i$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{n-4}{4}\right)^i+,\text{ etc.},$$

where n is the number of numbers and i the number of drawings. This is Meyer's answer to his Problem VIII, which reads: Eine Lotterie besteht aus n Nummern, in jeder Ziehung wird eine davon gezogen. Es soll die Wahrscheinlichkeit π gefunden werden, dass in i Ziehungen alle nummern erschienen sind.

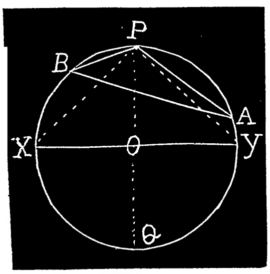
Professor Matz insists that the solution of Problem 158 in the May Number of the MONTHLY is also incorrect, his contention being that to take an arc for every point on AC and BD would not be taking them uniformly on AB. In reference to this contention, we must again call attention to Dr. Moore's *Note on Mean Value*, page 303, Vol. II of MONTHLY. The problem as stated is indefinite, and thus one may choose any law of distribution he wishes. Accordingly, the published solution is correct according to the law of distribution chosen. Other laws of distribution may be chosen, giving other results. ED. F.

161. Proposed by F. P. MATZ. Sc. D., Ph. D., Reading, Pa.

A triangle is inscribed at random in a circle; (a) what is the chance the triangle is *oblique;* and (b) what is the chance the triangle is *less in area* than $\frac{1}{4}\pi r^2$?

Solution by the PROPOSER.

(I) Put $OP=r$, $\angle APO=\theta$, $\angle BPO=\phi$; then $\triangle APB=2r^2\cos\theta\cos\phi\times\sin(\theta+\phi)$. In order that $\triangle APB$ may be *obtuse*-angled at P, the point A is restricted to the arc PAX, and the point B to the arc PBY. The required chance, therefore, becomes



$$C_1=\frac{2r^2\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\int_{\frac{1}{4}\pi}^{\phi}\cos\theta\cos\phi\sin(\theta+\phi)d\phi d\theta}{2r^2\int_0^{\frac{1}{2}\pi}\int_0^{\phi}\cos\theta\cos\phi\sin(\theta+\phi)d\phi d\theta}$$

$$=\frac{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}[2\sin^2\phi\cos^2\phi-\frac{1}{2}\cos^2\phi+(\phi-\frac{1}{4}\pi-\frac{1}{2})\sin\phi\cos\phi]d\phi}{\int_0^{\frac{1}{2}\pi}[2\sin^2\phi\cos^2\phi+\phi\sin\phi\cos\phi]d\phi}$$

$$=\left(\frac{\pi}{16}-\tfrac{1}{8}\right)/\frac{\pi}{4}=\tfrac{1}{2}\left(\tfrac{1}{2}-\frac{1}{\pi}\right).$$

(II) In this case the *superior* limit of ϕ in the numerator of C_1 is the value of ϕ derived from the equation $\sin\phi\cos^3\phi=\frac{1}{16}\pi$; and the *inferior* limit of the same variable is zero.

The required chance C_2 can, therefore, be found approximately; but is not of sufficient interest to warrant the labor required to find it.

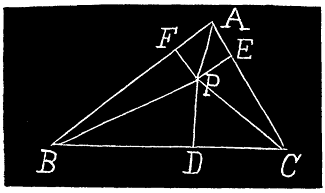
MISCELLANEOUS.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If P be a point within the scalene triangle, such that $\angle PAB=\angle PBC=\angle PCA=\phi$, then $\cot\phi=\cot A+\cot B+\cot C$.........(1), and $\operatorname{cosec}^2\phi=\operatorname{cosec}^2A+\operatorname{cosec}^2B+\operatorname{cosec}^2C$.........(2).

I. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Let $\angle PAB=\angle PCA=\angle PBC=\phi$. Then $\angle APB=\pi-(\phi+B-\phi)=\pi-B$. Draw PD, PE, PF perpendicular to BC, CA, AB, respectively.



$$PD=PB\sin\phi=\frac{AB\sin PAB}{\sin APB}.\sin\phi=\frac{c\sin^2\phi}{\sin B}$$

$$=\frac{2Rc}{b}\sin^2\phi.\quad\text{So }PE=2R\frac{a}{b}\sin^2\phi,$$

$$PF=2R\frac{b}{a}\sin^2\phi.$$

$$\frac{\sin(A-\phi)}{\sin\phi}=\frac{PE}{PF}=\frac{a^2}{bc}=\frac{\sin A\sin(B+C)}{\sin B\sin C}.$$

$\therefore\cot\phi-\cot A=\cot B+\cot C$, or $\cot\phi=\Sigma\cot A$.

Also $\cot^2\phi=\Sigma\cot^2A+2\Sigma\cot B\cot C$, $\operatorname{cosec}^2\phi-1=\Sigma\operatorname{cosec}^2A-3+2$; *i. e.*, $\operatorname{cosec}^2\phi=\Sigma\operatorname{cosec}^2A$.

II. Solution by the PROPOSER.

Let $PA=m$, $PB=n$, $PC=p$. $\sin(\pi-B):\sin(B-\phi)=c:m$.

$\therefore\cot\phi-\cot B=(m/c\sin\phi)=2\cot B$ ....(1).

Also, $\cot\phi-\cot C=(p/a\sin\phi)=2\cot C$..........(2);

and $\cot\phi-\cot A=(n/b\sin\phi)=2\cot A$.........(3).

Adding, and dividing by (3), we have $\cot\phi=\cot A+\cot B+\cot C$..........(A).

Squaring (A), and transforming into cosecants, we have

$$\operatorname{cosec}^2\phi=\operatorname{cosec}^2A+\operatorname{cosec}^2B+\operatorname{cosec}^2C.$$

Also solved by M. E. Graber, J. Scheffer, and A. H. Holmes.

GROUP THEORY.

7. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Which linear substitution will transform $x_1x_2+x_3x_4+x_5x_6=0$ into $y_1^2+y_2^2+y_3^2-y_4^2-y_5y_6=0$?

Remark by the PROPOSER.

One substitution having the desired property is

$$\left\{\begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6, \\ y_1+iy_2, & y_1-iy_2, & y_3+y_4, & y_3-y_4, & -y_5, & y_6, \end{matrix}\right\}^*$$

where $i=\sqrt{-1}$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

237. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading. Pa.

Solve $x^2+y+z=12$.....(1); $x+y^2+z=8$.....(2); $x+y+z^2=6$.....(3).

238. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{1}{1+n}+\frac{1}{3(n+3)}+\frac{1}{5(n+5)}+\ldots\ldots=\frac{1}{2}\left[\frac{1}{(n-1)}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\ldots+\frac{1}{l(n-l)}\right]$, n being an even positive integer and $l=n-1$.

239. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $\sqrt[4]{(41+x)}+\sqrt[4]{(41-x)}=4$.

GEOMETRY.

260. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Perpendiculars to the radius vector are drawn through points on $r=a+b\cos n\theta$. Find the radius of curvature of their envelope at a point at a given distance from the origin.

261. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Given three non-intersecting circles; to draw eight tangent circles, each tangent to all three of the given circles.

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*More generally, one set of substitutions fulfilling the required conditions, is

$$\left\{\begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6, \\ ay_1\pm aiy_2, & ay_1\mp aiy_2, & ay_3\pm ay_4, & ay_3\pm ay_4, & \pm ay_5, & \mp ay_6, \end{matrix}\right\}$$

where a is not equal 0. ED. E.

262. Proposed by NELSON L. RORAY, Utica, New York.

In a regular pentagon, show that $\frac{\text{diagonal}}{\text{side}}=\frac{2\text{ apothem}}{\text{radius}}=\frac{1}{2}.\frac{5+\sqrt{5}}{\sqrt{5}}$.

CALCULUS.

199. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the perimeter of an ellipse varies uniformly at the rate of $\frac{1}{4}$ inch per unit of time, at what rate is the eccentricity varying the instant the perimeter becomes 60 inches and the major axis 25 inches?

200. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Find the equation of a curve so that the area bounded by the curve, the axis of x, and any ordinate y, is equal to $y-x$, x being the *corresponding* abscissa.

201. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $\int\int^{dy/dx}\frac{dw}{1+w^2}=0$.

MECHANICS.

180. Proposed by EDWIN L. RICH, Lehigh University, South Bethlehem, Pa.

If a body is projected into the air, and the resistance of the air varies as the square of the velocity; required the equation of the curve. [From De Volson Wood's *Analytical Mechanics*, problem 10, p. 179.]

181. Proposed by F. ANDEREGG, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

A triangle AOB, of which the sides, OA, AB, and the angle at O are a, b, and α, revolves uniformly about O, so that OA makes the angle nt with the axis of x, and carries a circle of which AB is the diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe an ellipse whose axes are

$$\sqrt{(a^2+b^2+2ab\cos\alpha)}\pm\sqrt{(a^2+b^2-2ab\cos\alpha)}.$$

182. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

I have a tank, the lower part of which is a hemisphere 22 feet in diameter. The rest is a cylinder 22 feet in diameter, and altitude 28 feet. This tank is connected with the earth by a vertical stand-pipe 10 inches in diameter, 130 feet long, extending 2 feet into the tank. The tank is filled by a $2\frac{1}{2}$ inch pipe 65 feet long, having one right-angled elbow delivering the water into the bottom of the stand-pipe from a steam pump under 96 pounds gauge pressure. How long will it take to fill the pipe?

DIOPHANTINE ANALYSIS.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Can there be determined three cube numbers whose sum is the product of two squares?

AVERAGE AND PROBABILITY.

165. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square parallel to one of the diagonals?

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the average area intercepted by two non-intersecting chords drawn at random in a given circle.

167. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

GROUP THEORY.

8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the eight players being paired in each round, every pair to be matched once and but once in the tournament. List the possible programs different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

MISCELLANEOUS.

149. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Given $\sin^{-1}u+\sin^{-1}\frac{1}{2}u=\frac{1}{4}\pi$, to find u.

NOTES.

Dr. Saul Epsteen has been elected to the chair of Applied Mathematics in the University of Colorado, made vacant by the resignation of Dr. Arnold Emch who has accepted the professorship of Mathematics in his alma mater in Switzerland. F.

The April Number of the *Educational Times*, London, England, announced the death of Rev. Allen Whitworth, Prebendary of St. Paul's Cathedral. Mr. Whitworth was among the earliest contributors to the mathematical columns of the *Educational Times*, his name occurring in the list of contributors to the first volume of the Reprint. He was a recognized authority in pure probability, and his little book, "Choice and Chance," together with his "DCC Exercises" afford the most reliable source from which to gain information of the essential principles underlying the subject. F.

With this issue of the MONTHLY Dr. Saul Epsteen, who has so ably edited the Department of Problems and Solutions in the MONTHLY from October, 1903, to January, 1905, and who very kindly consented to assist in editing that Department since January, 1905, will retire from the editorial staff. It will be remembered that Dr. Epsteen assumed this work in order to relieve us from the care and responsibility while attending the University of Pennsylvania. We take this opportunity to express our keen appreciation of his services and to thank him for the interest, efficiency, and care he has taken in the work. F.

BOOKS AND PERIODICALS.

Supplementary Algebra. By R. L. Short. Small 8vo, Paper Cover. 45 pages. Boston: D. C. Heath & Co.

This monograph was written with a view to satisfy the demands coming from teachers for supplementary work in algebra. The graph is used to represent functions, and many of the illustrations have been taken from the calculus and mechanics. A number of excellent short methods in reduction and the solution of equations are introduced. F.

Examples in Algebra. Eight Thousand Exercises and Problems Carefully Graded from the Easiest to the Most Difficult. By Charles M. Clay, Head Master of Roxbury High School, Boston, Mass. 8vo, Cloth, viii+372 pages. Price, 90 cents. New York: The Macmillan Co.

The author sounds the key-note of successful algebra teaching, when he asserts that the first need of beginners in algebra is drill, drill, drill until the fundamental processes become as familiar to them as in arithmetic. He believes that too much time put in upon problems teaches the student to think rather than to act, and too much upon examples to act rather than to think, and so a judicious mixture of both is the only wise course. To this end these examples and problems are supplied. They are carefully graded and are well chosen. F.

Essentials of Algebra. Complete Course for Secondary Schools. By John C. Stone, A. M., Michigan State Normal School, and James F. Mills, A. M., The Shortridge High School, Indianapolis, Ind. 8vo, Half Leather, xii+412 pages. Boston, New York, Chicago: Benjamin F. Sanborn & Co.

Some of the special features of this book are, that the pupil is taught from the beginning how to check his work; he is required to see that every step in the solution of an equation is the application of a fundamental principle, "transposing" and "clearing of fractions" being used only after he thoroughly knows the underlying processes; the exercises, which are numerous and well graded; and the graph. The book is one which we believe will bear searching criticism. F.

Elementary Algebra. By Walter R. Marsh, Head Master Pingry School, Elizabeth, N. J. 8vo, Half Leather, vi+395 pages. New York: Charles Scribner's Sons.

This text follows the requirements of the College Entrance Examination Board as to subject matter, emphasis being placed upon those principles which are to be the tools of subsequent mathematical study. Many of the problems and illustrations are taken from physics and other practical sources. F.

The American Journal of Mathematics. Edited by Frank Morley and Simon Newcomb, and others. Published Quarterly under the auspices of the Johns Hopkins University.

The April Number contains the following articles: On a Class of Differential Equations, by A. Chessin; Surfaces with the Same Spherical Representation of their Lines of Curvature as Pseudo-Spherical Surfaces, by L. P. Eisenhart; On the Forms of Sextic Scrolls having no Rectilinear Directrix, by V. Snyder; Determination of the Ternary Modular Group, by L. E. Dickson. F.

The Annals of Mathematics. Published Quarterly under the auspices of Harvard University.

The April Number contains the following articles: Groups of the Fundamental Operations of Arithmetic, by G. A. Miller; Linear Differential Equations with Discontinuous Coefficients, by M. Bocher; Some Physical Solutions of the Equation of the nth Degree, by R. E. Moritz; The Ballistic Problem, by F. Gilman; A Theorem Concerning Uniform Convergence, by G. D. Birkhoff. F.

ERRATA.

Page 108, line 5, for "$(1.2.3.....a)^4\equiv0$" read $(1.2.3.....a)^4-1\equiv0$.
Page 136, line 9 from bottom, for "$x_1y_2-x_2y_2$" read $x_1y_2-x_2y_1$.

THE AMERICAN MATHEMATICAL MONTHLY.

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GROUPS CONTAINING THE LARGEST POSSIBLE NUMBER OF OPERATORS OF ORDER TWO.

By G. A. MILLER.

The abelian group of order 2^m and of type (1, 1, 1, ...) contains 2^m-1 operators of order two. This group will not be considered in what follows. Let G represent any other group whose order (g) is divisible by the prime number p. We proceed to prove that the number of the operators of G whose orders exceed two is at least $\frac{p-1}{2p}g$ and to determine all the possible groups in which the number of these operators is exactly $\frac{p-1}{2p}g$. The considerations are extremely simple but the results seem of sufficient importance to be given in an explicit form.

If g is divisible by 2 but not by 4 it is well known that G contains a subgroup (H) of half its order which is composed of its operators of odd order in addition to the identity. In such a group the number of operators of order 2 cannot exceed $g/2$, and whenever the number of operators of order two contained in G is exactly $g/2$ then the order of G is twice an odd number and the subgroup H is abelian. These known facts are stated here merely for the sake of completeness. They are not essential in what follows.

If g is divisible by 4 it is clearly possible to construct a G such that more than half its operators are of order 2. Such a group is obtained by extending the cyclic group of order $g/2$ by the operators of order two which transform each operator of this cyclic group into its inverse. The group obtained in this man-

ner is known as the dihedral rotation group of order g. In what follows it will be assumed that g is divisible by 4, and that more than half the operators of G are of order two. As it is assumed that G contains operators whose orders exceed two and as G is generated by its operators of order two, it follows that G is non-abelian and contains non-invariant operators of order 2.

Let t represent any non-invariant operator of order 2 contained in G. The subgroup (K) of G which is composed of all its operators which are commutative with t must include t. Let $G-K$ represent all the operators of G which are not also in K. All the products obtained by multiplying t into operators of $G-K$ are found in $G-K$. Since t is not commutative with any of these operators, the products obtained by multiplying t into any operator of order 2 in $G-K$ are of orders greater than 2. That is, $G-K$ contains at least as many operators whose orders exceed 2 as of order 2. Hence more than half the operators of K are of order 2.

If $G-K$ contains an operator of order p the index of K under G cannot be less than p.* In this case the number of operators whose orders exceed p in $G-K$ is not less than $\frac{p-1}{2p}g$ since the total number of operators in $G-K$ is not less than $\frac{p-1}{p}g$. If $G-K$ does not contain an operator of order p such an operator must occur in K as g is divisible by p. In this case, $G-K$ cannot contain less than $g/2-g/2l$ operators whose orders exceed 2, l being the index of K under G.

As the order of K is less than that of G and as K satisfies the conditions which G was assumed to satisfy whenever K includes an operator of order p,† we shall for the present suppose that the theorem in question is true with respect to K. That is, K contains at least $\frac{p-1}{2p}.g/l$ operators whose orders exceed two. In this case G must contain at least $g/2-g/2l+\frac{p-1}{2p}.g/l=\frac{p\,l-1}{2p\,l}g$ operators whose orders exceed two. As this number exceeds $\frac{p-1}{2p}g$, the theorem in question is true with respect to G provided it is true with respect to K. As it is evidently true with $4p$, it is universally true.

Having proved that the number of operators whose orders exceed 2 is at least equal to $\frac{p-1}{2p}g$, it remains to determine all the possible groups of order g in which there are exactly $\frac{p-1}{2p}g$ operators whose orders exceed two. We shall

*This follows from the usual rectangular arrangement of the operators of G, the operators of K constituting the first row of the rectangle. The theorem is included in the following known theorem: If a group G contains a subgroup K of index l, and if n is the lowest power of some operator (s) of G such that s^n is contained in K, then $n<l+1$.

†The order of K is divisible by 4 since it contains an invariant operator of order two and other operators of this order.

arrive at the remarkable result that *all such groups are the direct products of the dihedral rotation group of order* $4p$ *and an abelian group of order* 2^{α} *and of type* (1, 1, 1, ...). When $p>2$ such a group is also the direct product of the dihedral rotation group of order $2p$ and an abelian group of order $2^{\alpha-1}$ and of type (1, 1, 1, ...). We proceed to prove these results.

It has been proved above that K is the abelian group of order 2^{β} and of type (1, 1, 1, ...) whenever G contains a minimum number of operators whose orders exceed two. As this must be true of every K for the different non-invariant operators of order two contained in G and as the index of all of these K's is p, it follows that each non-invariant operator of order two contained in G has just p conjugates under G. Since any K is maximal it follows that the operators which are common to two K's are invariant under G, and vice versa. We shall soon see that these common operators include just half of all the operators of G.

As all the products obtained by multiplying t into operators whose orders exceed two are of order two, t must transform every operator whose order exceeds two into its inverse. Hence every non-invariant operator of order two contained in G must have this property. These operators constitute half the operators of $G-K$ and at least half the operators of K. Since they could not constitute more than half of the operators of G (for there are just as many operators that are commutative with any operator in G as there are of those which transform this operator into its inverse) it results that just half the operators of K are invariant under G. These operators constitute an invariant subgroup K' of order $g/2p$.

When $p>2$, G includes an invariant subgroup of order $2p$, since an operator of order p is transformed either into itself or into its inverse by all the operators of G. In this case G is the direct product of K' and this subgroup, the dihedral rotation group of order $2p$, since these two invariant subgroups have only the identity in common and the product of their orders is g*. When $p=2$; that is, when $g=2^m$, K is invariant.† Half of the operators of $G-K$ are of order 4. Hence G includes the octic group as an invariant subgroup. As this has two operators in common with K' such a G is the direct product of the octic group and any subgroup of half the order of K' and included in K' which does not include the commutator of order 2. This completes the proof of the theorems in question; and, as these direct products are so elementary, it seems unnecessary to go into further details.

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*Cf. Burnside, Theory of Groups of Finite Order, 1897, p. 44.

†In all other cases K has p conjugates under G.

ON THE MOTION OF A BALL ON A BILLIARD TABLE.

By N. J. LENNES, University of Chicago.

A ball, of radius r, is assumed to move in straight lines and without friction on a rectangular billiard-table, the lengths of whose sides are a and b. Under the assumption that the angle of reflection is equal to the angle of incidence, it is required to discuss completely the motion of the center of the ball when started from a given point and in a given direction.

Under the conditions of the problem the center of the ball must always remain at least at a distance r from the sides of the table, *i. e.*, the center cannot move outside a rectangle whose sides are $a-2r$ and $b-2r$. The problem is obviously equivalent to the problem in which a ball of zero radius is considered to move on a table of dimensions $a-2r$ and $b-2r$. Denote $a-2r$ and $b-2r$ by m and n, and denote the corners of this last rectangle in their cyclic order by A, B, C, D. When we say that the ball strikes one of the sides of the table, we mean that the center of the ball is on one of the segments AB, BC, CD, DA. When the ball strikes a corner of the table the center is at A, B, C, or D. The angle at which the ball strikes the side of the table is the angle between the line of motion and the perpendicular to that side.

The following propositions are immediate consequences of the conditions of the problem:

(1). If the ball is reflected back and forth several times between two opposite sides without striking the remaining sides then the points at which it strikes one side are equidistant and all the lines of motion from either one of these sides to the other are parallel.

(2). If the ball is reflected from one side, as BC, to an adjacent side, as CD, the line of reflection from the side CD is parallel to the line of incidence with BC.

(3). If the ball strikes a corner it is reflected back along the line on which it approached the corner.

(4). If the ball is reflected from a side, as BC, to an adjacent side CD, then it strikes the side CD the same distance from the point C that it would if the table were so extended that the ball would move freely in the direction in which it moved before it struck BC until it should meet the side CD extended.

It follows directly from (3) that if the ball strikes a corner it will retrace its path. These considerations enable us to investigate the periodicity of the motion by considering the problem in the following simplified form, viz., we consider a table in which one side is removed and the table extended indefinitely in that direction. Let the side removed be one whose length is m. On the two extended sides lay off the distance n an indefinite number of times beginning at the corners of the original table. These points are referred to as multiple points of n. The motion of the ball on the original table may now be regarded as transformed into the motion of the ball on the extended table. It is quite easy

to see, from (1)—(4), that if the ball strikes one of the multiple points of n on the extended table then in the corresponding motion on the original table it would strike a corner, and conversely if the ball strikes a corner on the original table then it strikes a multiple point of n on the extended table.

Suppose first that the ball is started from the corner A in a direction whose angle with AB is α. Let the side CD opposite AB be removed and the table extended indefinitely in that direction. Under what conditions will the ball strike a multiple point of n? The answer is: If $m\tan\alpha$ and n are commensurable, the length of AB being m and that of BC being n. If $m\tan\alpha$ and n have a common measure l, which is contained h times in $m\tan\alpha$ and k times in n, then $km\tan\alpha = hn$. Consequently the ball will strike a multiple point of n at a distance hn from the side AB. We may assume that l is the greatest common measure of $m\tan\alpha$ and n. Hence h and k are integers prime to each other. It follows that hn will be the first multiple point of n which the ball strikes. If k is an even number this multiple point of n will be on the side AD extended (the side from which the ball was started).

In every case when the ball strikes the first multiple point of n on the side AD extended, h is odd; for h and k being prime to each other cannot both be even. By direct consideration of the phenomenon on the table this appears from the fact that if h were even we should have by the symmetry of the motion that the ball would strike an odd multiple of n on the side BC half way between the side AB and the multiple point of n in question. This corresponds to the fact that on the original table the ball cannot return to the point A without returning along the path on which it left A; that it cannot strike A without first striking another corner. If k is odd the first multiple point of n reached is on the side BC extended. If h and k are both odd, then the ball will strike the side BC extended in an odd multiple point of n, which means that in the original motion it would strike the corner C. If k is odd and h even the ball will strike the side BC in an even multiple point of n, which corresponds to the point n on the original table.

If k is even the ball will strike each of the sides AD and BC $\frac{1}{2}k$ times in a half period (counting the starting point A but not the point D). If k is odd, it strikes each of the sides AD and BC $\frac{1}{2}(k+1)$ times (in this case counting both the starting point and the point of final impact). Similarly, if h is even the ball will strike each of the sides AB and CD $\frac{1}{2}h$ times counting the starting point but not the point of final impact, and if h is odd it strikes each of the sides AB and CD $\frac{1}{2}(h+1)$ times (counting both the point of departure and the point of final impact).

If k is even and h is odd the first half period ends at D, if h is even and k is odd it ends at B, and if both h and k are odd it ends at C.

If the ball is started from a point P, on the side AD, P being different from A, there are four possibilities, viz., (a) $m\tan\alpha$ is commensurable with n but not with AP or PD. (b) $m\tan\alpha$ is commensurable with n and with either AP or PD (in this case it follows as a theorem that $m\tan\alpha$ is commensurable

with both AP and PD). (c) $m\tan\alpha$ incommensurable with n but commensurable with one of the segments AP and PD (in this case it is a theorem that $m\tan\alpha$ is incommensurable with one of the segments AP and PD). (d) $m\tan\alpha$ incommensurable with n and with both of the segments AP and PD. We consider each case separately.

(a) If $m\tan\alpha$ is commensurable with n but not with AP or PD, then obviously the ball cannot strike a multiple point of n and hence not a corner on the original table but will strike points whose distance from a multiple point of n is AP. If h and k are integers such that $km\tan\alpha = hn$, and if k is even the ball will strike the side AD at a point whose distance from A is $AP+hn$. (We denote by P' all points whose distance from A is $AP+In$ where I is any integer). This corresponds to striking the point P on the original table. Since the ball cannot reach P along the line on which it was started from P but must reach it in a direction whose angle with the side AD is equal to that at which it left P it follows that the ball completes a period on reaching P.

If k is odd the ball will strike a point P' on the side AD which again corresponds to striking P on the closed table.

There is no half period. The number of impacts against each of the sides AD and BC is $\frac{1}{2}k$ if k is even (counting the point P once), and if k is odd (again counting the P once). Similarly, it strikes each of the sides AB and CD $\frac{1}{2}h$ times if k is even, and h times if k is odd.

(b) Under (b) two cases are possible, viz., (1) The ball may strike a point P' on the side AD before it strikes a multiple point of n. The result is the same as in (a). (2) If the ball strikes a multiple point of n before it strikes a point P' it comes under the case discussed above where the ball is started from a corner.

(c) In case $m\tan\alpha$ if incommensurable with n but commensurable with one of the segments AP and PD, then obviously the ball cannot reach a point P'. Further, it can reach at most one corner on the original table; consequently the motion is not periodic in this case.

(d) In case $m\tan\alpha$ is incommensurable with n and both segments AP and PD the motion is obviously not periodic.

In case the motion is not periodic every point on the table lies in the path of the ball or is a limit point of such points. This can be proved by means of the following lemma: *Given any two irrational numbers a and b and any positive number e, however small, then it is possible* to find a pair of integers A and B such that* $|Aa+Bb| < e$.

Consider now the case where the ball started from A. According to the lemma for any preassigned positive number e there exists a pair of integers h and k such that $|km\tan\alpha - hn| < e$. This shows that on the extended table the ball comes as near as we please to multiple points of n. If k is even, D is obviously a limit point of the set of points passed over by the ball. In the same manner we can show that any point on either one of the sides AD or BC

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*Jules Tannery, *Theorie des Fonctions*, 2d ed., I, p. 37.

which is not reached by the ball is a limit point of points which it does reach. Since the paths of the ball from one side to the opposite side are parallel it follows that every point on the table not reached by the ball is a limit point of points which it does reach.

Evidently the same conclusion follows in the non-periodic cases in which the ball starts from a point other than a corner of the table.

DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

SOLUTIONS OF PROBLEMS.

NOTE. The following solutions were received too late for publication: Calculus, No. 196, solved by J. Scheffer, and A. H. Holmes; No. 197, solved and discussed by Miss Ida M. Schottenfels. Geometry, No. 255, solved by William Hoover; No. 257, solved by F. R. Honey. Mechanics, No. 179, solved by William Hoover, and J. Scheffer.

ALGEBRA.

234. Proposed by G. W. GREENWOOD. M. A. (Oxon). Professor of Mathematics and Astronomy, McKendree College. Lebanon, Ill.

Prove that $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\ldots\ldots\ldots=n!$

Solution by the PROPOSER.

$\triangle u_x=u_{x+1}-u_x=(E-1)u_x$ where $Eu_x=u_{x+1}$.

Hence $(E-1)x^n=\triangle x^n$, $(E-1)^n x^n=\triangle^n x^n$, *i. e.*, $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\ldots\ldots=\triangle^n x^n!=n!$ by a well known theorem in the Calculus of Finite Differences.*

Also solved by J. Scheffer.

235. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Easter Sunday, 1905, was on April 23. How often in the last one hundred years has this occurred, and when?†

Solution by the PROPOSER.

The Dominical Letter for April 23, 1905, is A. No other year thus far in the twentieth century had A for its Dominical Letter.

*Epsteen, MONTHLY, June-July, 1904, p. 132.

†Dr. Zerr adds the remark that this will occur again in 1916 and 2000.

The years of the nineteenth century having A for their Dominical Letter were: 1809, 1815, 1820, 1826, 1837, 1843, 1848, 1854, 1865, 1871, 1876, 1882, 1893, the Epacts for which are 14, 20, 15, 22, 23, 0, 25, 1, 3, 9, 4, 11, 12.

Of these the years admissible must have the Epacts 24, 25, 26, 27, one only meeting the conditions, viz., 1848.

Also solved by J. Scheffer, and A. H. Holmes.

236. Proposed by L. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

Sum to infinity the series $\frac{n^2}{(n+1)!}$ beginning with $n=1$.

Solution by L. E. NEWCOMB, Los Gatos, Cal.

The general term $\frac{n^2}{(n+1)!}=n\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)$. Set $n=1, 2, 3, \ldots$ in succession, and the series becomes

$$1-\frac{1}{2}+\frac{2}{2!}-\frac{2}{3!}+\frac{3}{3!}-\frac{3}{4!}\ldots\ldots=1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots=e-1.$$

Also solved by G. W. Greenwood, J. Scheffer, and the Proposer.

AVERAGE AND PROBABILITY.

162. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Two points are taken at random in the surface of a circle and a chord is drawn through them. Find the average area of the segment containing the center of the circle.

Solution by the PROPOSER.

Let 2θ=the sectoral angle, then the area of the segment in question is $U=\frac{1}{2}(2\pi-2\theta+\frac{1}{2}\sin 2\theta)r^2$.

$$\therefore A=\frac{r^2}{\pi}\int_0^{\frac{1}{2}\pi}Ud\theta=\tfrac{1}{2}\left(\frac{3}{2}\pi+\frac{1}{\pi}\right)r^2.$$

CALCULUS.

199. Proposed by F. P. MATZ. Sc. D., Ph. D., Reading, Pa.

If the perimeter of an ellipse varies uniformly at the rate of $\frac{1}{4}$ inch per unit of time, at what rate is the eccentricity varying the instant the perimeter becomes 60 inches and the major axis 25 inches?

No correct solution for this problem has been received.

200. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Find the equation of a curve so that the area bounded by the curve, the axis of x, and any ordinate y, is equal to $y-x$, x being the *corresponding* abscissa.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^x ydx=y-x. \quad \therefore ydx=dy-dx, \text{ or } dx=\frac{dy}{y+1}.$$

$x=\log(y+1)$, and $e^x=(y+1)$ is the required equation.

Also solved by S. A. Corey, F. P. Matz, and the Proposer.

DIOPHANTINE ANALYSIS.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Can there be determined three cube numbers whose sum is the product of two squares?

I. Solution by A. H. HOLMES, Brunswick, Maine.

Take x^3, x^3y^3, and x^3z^3, for the three cube numbers, and u^2 and v^2 for the square numbers.

$\therefore x^3(1+y^3+z^3)=u^2v^2$. If we put $y=2$, $z=3$, $1+2^3+3^3=6^2=u^2$, $x^3=v^2=w^6$. $\therefore x=w^2$, $v=w^3$. Put $w=2$. $\therefore x=4$, $v=8$.

$\therefore 64+512+1728=36\times 64=2304$, or $4^3+8^3+12^3=6^2\times 8^2$.

If it is required that no two of the numbers to be cubed and squared should be the same, put $x^3=uv$, so that $1+y^3+z^3=uv$.

Put $x=9$, $y=6$, and $z=8$. Then $u=3$, $v=243$.

$\therefore 9^3+54^3+72^3=3^2\times 243^2$.

II. Solution by J. EDWARD SANDERS, Hackney, Ohio.

If $a^3+b^3+c^3=d^3=(e.f)^3$, then one set of numbers satisfying the condition is: $(a.d^{2n-1})^3+(b.d^{2n-1})^3+(c.d^{2n-1})^3=d^{6n}=(e^{3n})^2.(f^{3n})^2$.

Again, since $1^3+2^3+3^3=2^2.3^2$, another solution is $(a^4)^3+(2a^4)^3+(3a^4)^3=(2a^3)^2.(3a^3)^2$.

Also solved by G. B. M. Zerr, R. D. Carmichael, and the Proposer.

GEOMETRY.

NOTE. Problems 259 and 261 are identical. A solution may be found in Lachlan's *Modern Pure Geometry*, pp. 241-2.

254. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the cartesian equation of a line that is both tangent and normal to the cardioid.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let the required line be $2ax+2by+c=0$, which is to be both tangent and normal to $(x^2+y^2+cx)^2=c^2(x^2+y^2)$.

The constants a and b are to be found so as to satisfy the given conditions.

Transform the line and cardioid through the inversion

$$x=\frac{cx'}{x'^2+y'^2},\quad y=\frac{cy'}{x'^2+y'^2}$$

obtaining, after dropping accents,

$$x^2+y^2+2ax+2by=0\,;\ y^2=2x+1.$$

Since angles are not changed by inversion the conditions to be satisfied are now that the circle shall be both tangent and normal to the parabola.

The latter condition requires that the coördinates of one intersection shall satisfy $y^2+by-x-a=0$.

Elimination of y^2 from these three equations followed by elimination of y from the two equations thus obtained gives $(x+1)(x+2a-1)=0$.

The first root is not used as it leads to an imaginary intersection.

From the second root the intersection is

$$[1-2a,\ \pm\sqrt{(3-4a)}],\ \text{also}\ b=\frac{3a-2}{\pm\sqrt{(3-4a)}}.$$

With this value of b the abscissae of intersections of circle and parabola are given by

$$(x^2+2ax+2x+1)^2-\frac{4(3a-2)}{3-4a}(2x+1)=0.$$

Removing the factor $x+2a+1$ which corresponds to the orthogonal intersection, the result is the following cubic,

$$x^3+x^2(2a+5)+11x+\frac{13-18\,a}{3-4\,a}=0.$$

As the circle is to be tangent to the parabola this equation must have a double root, *i. e.*, its discriminant must vanish, thus giving the following equation for the determination of a,

$$9a^5+24a^4-143a^3+202a^2-116a+24=0.$$

The roots of the latter are 1, 1, $\frac{2}{3}$, $\frac{2}{3}$, -6, of which the first four lead to imaginary intersections.

When a is -6, b is $\dfrac{\mp 20\sqrt{3}}{9}$, so that the desired result is*

$$108x\pm 40\sqrt{3}y-9c=0.$$

*Dr. Hoover obtains the obvious line $y=0$, but not the ones of the published solution. Dr. Safford's fortunate choice of c for absolute term in the assumed line excludes the line $y=0$ from his result. A perfectly general method would furnish a singly infinite family of lines only the above three of which are real. An investigation of this family for, say the general nodal-circular quartic, ought to lead to some interesting results. Ed. G.

262. Proposed by NELSON L. RORAY, Utica, New York.

In a regular pentagon, show that $\dfrac{\text{diagonal}}{\text{side}}=\dfrac{2\text{ apothem}}{\text{radius}}=\frac{1}{2}.\dfrac{5+\sqrt{5}}{\sqrt{5}}$.

Solution by P. S. BERG, Larimore, N. Dak.

In a regular pentagon the diagonal is $\frac{R}{2}\sqrt{(10+2\sqrt{5})}$, and the side is $\frac{R}{2}\sqrt{(10-2\sqrt{5})}$.

$$\frac{\frac{R}{2}\sqrt{(10+2\sqrt{5})}}{\frac{R}{2}\sqrt{(10-2\sqrt{5})}}=\frac{\frac{R}{2}\sqrt{(6+2\sqrt{5})}}{R}.$$

But $\frac{R}{2}\sqrt{(6+2\sqrt{5})}=2$ apothem, and $R=$radius.

$$\frac{\frac{R}{2}\sqrt{(6+2\sqrt{5})}}{R}=\frac{\sqrt{(6+2\sqrt{5})}}{2}=\frac{1}{2}.\frac{\sqrt{(30+10\sqrt{5})}}{\sqrt{5}}=\frac{1}{2}.\frac{5+\sqrt{5}}{\sqrt{5}}.$$

Also solved by G. B. M. Zerr, R. D. Carmichael, A. H. Holmes, and L. E. Newcomb.

MECHANICS.

180. Proposed by EDWIN L. RICH, Lehigh University, South Bethlehem, Pa.

If a body is projected into the air, and the resistance of the air varies as the square of the velocity; required the equation of the curve. [From De Volson Wood's *Analytical Mechanics*, problem 10, p. 179.]

Solution by G. B. M. ZERR.

Let the direction of projection be in the xy plane. Let v=velocity of projection, g=acceleration of gravity, α=angle of projection, $k(ds/dt)^2$=resistance at any time t, ϕ=inclination of direction of motion to the horizon at any time t. The x- and y- components of resistance are, respectively, $k\dfrac{ds}{dt}\dfrac{dx}{dt}$, and $k\dfrac{ds}{dt}\dfrac{dy}{dt}$.

Resolving horizontally and vertically, the equations of motion are

$$d^2x/dt^2=-k(ds/dt)(dx/dt).....(1),$$
$$dy^2/dt^2=-g-k(ds/dt)(dy/dt).....(2).$$

From (1), $d(dx/dt)/(dx/dt)=-kds$.

$\therefore \log[(dx/dt)/v\cos\alpha]=-ks$. When $t=0$, $dx/dt=v\cos\alpha$.

$\therefore dx/dt=v\cos\alpha e^{-ks}=u.....(3)$.

Resolving in the direction of the tangent and normal,

$$d^2s/dt^2=-g\sin\varphi-k(ds/dt)^2 \ldots\ldots(4).$$

Let v_1 =velocity of particle at any point; then (4) becomes

$$d^2s/dt^2=-g\sin\phi-kv_1^2 \ldots\ldots(5),$$
$$v_1^2/\rho=g\cos\phi \ldots\ldots(6).$$

But $u=v_1\cos\varphi$ and $\rho=-ds/d\phi$. $\therefore$ (5) and (6) become

$$du/dt=-kv_1^2\cos\phi \ldots\ldots(7),$$
$$v_1(d\phi/dt)=-g\cos\phi \ldots\ldots(8).$$

$$\therefore \frac{du}{d\phi}=\frac{k\,v_1^3}{g}=\frac{k}{g}u^3\sec^3\phi \ldots\ldots(9).$$

At the origin, $u=v\cos\alpha$. Integrating (9) we get

$$\frac{1}{v^2\cos^2\alpha}-\frac{1}{u^2}=\frac{k}{g}(A_\phi-A_\alpha) \ldots\ldots(10).$$

Where $A_\phi=\int_0^\phi 2\sec^3\phi d\phi$, $A_\alpha-A_\phi=2\int_\phi^\alpha \sec^3\phi d\phi$

$$=\tan\alpha\sec\alpha-\tan\phi\sec\phi+\log\left(\frac{\tan\alpha+\sec\alpha}{\tan\phi+\sec\phi}\right).$$

bstituting in (10) we get

$$\frac{k}{g}v^2\cos^2\alpha\left[\tan\alpha\sec\alpha-\tan\phi\sec\phi+\log\left(\frac{\tan\alpha+\sec\alpha}{\tan\phi+\sec\phi}\right)\right]=e^{2ks}-1,$$

for the intrinsic equation to the curve.

MISCELLANEOUS.

148. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Given $\sin 3\phi+\cos 3\phi=m$.............(1), and $\cos\phi-\sin\phi=x$...............(2), to find x in terms of m.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $3\sin\phi-4\sin^3\phi=\sin 3\phi$ and $-3\cos\phi+4\cos^3\phi=\cos 3\phi$, we have $4(\cos^3\phi-\sin^3\phi)-3(\cos\phi-\sin\phi)=m$, or $(\cos\phi-\sin\phi)[4(\cos^2\phi+\cos\phi\sin\phi+\sin^2\phi)-3]=m$.

Since $\cos\phi-\sin\phi=x$, $2\sin\phi\cos\phi=1-x^2$ and therefore $2x^3-3x+m=0$.

Also solved by A. H. Holmes, and the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

240. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $a^2x+b^2y=ax^2+by^2=x^3+y^3$.

241. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Sum to infinity $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{840}+\dots$

242. Proposed by DR. L. E. DICKSON, The University of Chicago.

If $u_0{}^i h_0+u_1{}^i h_1+\dots u_r{}^i h_r=0$ $(i=0, 1, \dots, r-1)$, then $h_i \prod_{\substack{j=0,1,\dots,r \\ j \text{ not } = \text{ to } i}} (u_i-u_j)=0$.

243. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Find the infinite root of $\frac{1}{x}+\frac{1}{a}=\sqrt{\left[\frac{1}{a^2}-\sqrt{\frac{1}{a^2x^2}+\frac{1}{x^4}}\right]}$.

AVERAGE AND PROBABILITY.

168. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average area of a triangle two of whose sides have the constant sum $2a$.

169. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square?

170. Proposed by LON C. WALKER, Santa Barbara, Cal.

Find the area of a triangle formed by drawing a line at random through each of the three points taken at random within the surface of a given triangle.

171. Proposed by O. E. GLENN, A. M., Ph. D., Drury College.

There are n derelict steamers afloat in a circular sea of radius r. The water in the sea is moving northward in a current whose velocity varies inversely as the perpendicular distance from the north-south tangent to the sea on its west beach. Find the probability that a ship crossing the sea on a random diameter will encounter e derelicts during the voyage.

CALCULUS.

202. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the complete primitive of $y=2px+ap^2$. Regard the primitive as the equation giving the arbitrary constant, and if the primitive has equal roots discuss the equation expressing that condition.

203. Proposed by S. A. COREY, Hiteman, Iowa.

$$\int_0^\pi \frac{\sin mx}{x} dx,\ m=\text{integer}.$$

204. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Required the variation of $\int V dx$ where V is a function of $x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}$..... and v where $v=\int V' dx$ and V' is also a function of $x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}$.....

205. Proposed by Z. T. JACKSON, St. Louis, Mo.

Evaluate $\int_0^{\frac{1}{2}\pi} \log \sin x\, dx$.

206. Proposed by DR. O. E. GLENN, Drury College.

Evaluate $\int_0^1 (1-z^n)^m \frac{\partial}{\partial z} \log(1-z^n x^n) dz$, assuming $-1<x^n<+1$.

DIOPHANTINE ANALYSIS.

128. Prpposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Required the highest powers of 2, 3, 5, 7, contained in (1000)!

129. Proposed by SYLVESTER ROBINS.

How many perfect squares containing 2^n figures each can be found, the parts of which standing on the right hand side thereof, represented by 1, 2, 4, 8, 16, 32, etc., digits, are also perfect squares. 24591681 is one such number. [From *The Mathematical Visitor*].

GEOMETRY.

263. Proposed by FREDERICK R. HONEY, Trinity College, Hartford, Conn.

Construct a sphere whose surface shall intersect the surface of any four given spheres in great circles.

264. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

Let l and m be two straight lines intersecting in A. With A as center and any radius r describe a circle intersecting l and m in E, M and G, Q, respectively, and the bisector of the opposite angles formed by l and m in F and K. With I, the middle point of EA, as center, and radius, r, describe an arc intersecting the bisector of the opposite angles formed by l and m in O. With O as center, and radius $OA+r$ describe circle $FHCDBJF$, F and D the points of intersection of this circle with the bisector of opposite angle; H, B the intersections on l, and J, C on m. What is the ratio of arc HFJ to arc BD?

265. Proposed by R. D. CARMICHAEL, Hartselle. Ala.

Find the Cartesian equation of a curve in a vertical plane such that a particle, sliding down the curve under the force of gravity alone, will require to pass from any point of beginning to the lowest point of the curve, a time proportional to the square of the distance to be traversed along the curve.

266. Proposed by DR. O. E. GLENN.

Given the feet of the three perpendiculars from any point a on the circumcircle to the sides of the triangle are collinear, then if on the three chords $\overline{ab}$, $\overline{ac}$, $\overline{ad}$, as diameters circles be described, the points of intersection of these circles are collinear. [Salmon's *Higher Plane Curves*].

267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles in a plane is an isothermal system.

GROUP THEORY.

9. Proposed by DR. L. E. DICKSON, The University of Chicago.

Does there exist a triply transitive group on m letters of order $m(m-1)(m-2)$ other than the linear fractional group in the Galois Field of order $p^n=m-1$ and the group 720_3 on 10 letters (Cole, *Quarterly Journal*, 1895, p. 44)? The question relates to Problem 99, MONTHLY, March, 1900.

10. Proposed by DR. O. E. GLENN.

Find the order of the group of isomorphisms of the group of order p^4 defined by the relations $P_1^{p^2}=P_2^{p^2}=I$, $P_1P_2=P_2P_1$.

MECHANICS.

183. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two light rods are jointed at O. OA is attached by a hinge at A to a fixed rod AB and B is attached to a ring which can slide along AD. A force P acts at O towards AB at right angles to AB, and force Q acts at B along BA. The angles OAB, OBA are acute. There is no friction in the system. Show that for equilibrium we must have $P/Q=\sin AOB/(\cos AOB.\cos OBA)$.

184. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere, radius a, rests between two parallel thin perfectly rough rods A and B in the same horizontal plane at a distance apart equal to $2c$; the sphere is turned about A until its center is very nearly vertically over A; it is then allowed to fall back. Prove that it will rock between A and B if $10c^2<7a^2$; also, that θ_r, the angle through which it will turn after the rth impact is given by the equation $\cos\theta_r=\dfrac{\sqrt{(a^2-c^2)}}{a}+\dfrac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\dfrac{10c^2}{7a^2}\right)^{2r}$.

MISCELLANEOUS.

150. Proposed by T. N. HAUN, Mohawk, Tenn.

If $\dfrac{\sin\phi}{\sin\psi}=m$, find maximum and minimum value of $\dfrac{\sin(\phi+\theta)}{\sin(\psi+\theta)}$, where θ is known.

151. Proposed by W. J. GREENSTREET.

Sum the series $\sum\limits_{r=1}^{r=m}\operatorname{cosec}\left(\dfrac{2r-1}{4m}\pi+\theta\right)\operatorname{cosec}\left(\dfrac{2r-1}{4m}\pi-\theta\right)$.

NOTES.

Mr. A. R. Crathorne of the University of Wisconsin is a student at Göttingen.

Professor W. D. Cairns of Oberlin College is spending the year in research at Göttingen.

Professor H. S. White of Northwestern University has been called to Vassar College.

Mr. W. C. Brenke has been appointed Austin Teaching Fellow in Astronomy at Harvard.

Mr. T. C. Jones has been appointed fellow in mathematics at the University of Pennsylvania.

Mr. H. W. Reddick has been appointed fellow in mathematics at the University of Illinois.

Dr. W. D. Westfall has been appointed instructor in mathematics at the University of Missouri.

Mr. Homer R. Higley has been appointed instructor in mathematics in the Pennsylvania State College.

Dr. W. B. Ford of Williams College has been appointed instructor in mathematics at the University of Michigan.

The assistants in mathematics for 1905-6 at Indiana University are Mr. Charles Haseman, and Mr. D. J. Crittenberger.

The University of Washington has appointed Professor F. M. Morrison of Buchtel College as assistant professor of mathematics.

Professor H. F. Blichfeldt will spend next year in research at European universities, on leave of absence from Stanford University.

Professor G. A. Bliss, of the University of Missouri, has been appointed assistant professor of mathematics at Princeton University.

At the University of Indiana Dr. S. C. Davisson, and Dr. David A. Rothrock have been promoted to junior professorships in mathematics.

Dr. L. I. Neikirk, formerly Harrison Research Fellow at the University of Pennsylvania, has been appointed instructor in mathematics at the University of Illinois.

Mr. Louis Fussell, Lippincott Fellow in Mathematical Physics at Swarthmore College, is spending the year in graduate study at the University of Wisconsin.

Professor B. F. Finkel has been appointed Harrison Fellow in mathematics at the University of Pennsylvania, and is spending the year in graduate study in Philadelphia.

The Springfield Section of the Missouri Society of Teachers of Mathematics has effected organization for the year with a membership of twenty-five. The executive committee in charge of the work consists of Dr. O. E. Glenn, Miss Elizabeth Park, and Miss Nena Baxter.

The July number of the *Bulletin of the American Mathematical Society* contains Professor Henry Dallas Thompson's translation of M. Gaston Darboux's address, "A Survey of the Development of Geometric Methods." The address was delivered at St. Louis, September 24, 1904, before the geometry section of the International Congress of Arts and Science.

The Bolyai prize, established by the Hungarian Academy of Sciences in memory of John Bolyai and his father Farkas Bolyai, will be awarded this year for the first time. The awarding commission will deliberate in Budapest in October. It consists of the following members: Gaston Darboux (Paris), Felix Klein (Göttingen), Julius König (Budapest), Gustav Rados (Budapest).

The prize consists of a medal and ten thousand crowns, and will be awarded to the author of the best work in mathematics published during the five years preceding.

The current number of *Proceedings of the London Mathematical Society* contains the following papers: On the Reducibility of Covariants of Binary Quantics of Infinite Order, by Mr. P. W. Wood; Alternative Expressions for Perpetuant Type Forms, by Mr. P. W. Wood; Theorems on the Logarithmic Potential, by Prof. T. J. I'A. Bromwich; Ordinary Inner Limiting Sets in the Plane or Higher Space, by Dr. W. H. Young; A Method for Determining the Behaviour of Certain Classes of Power Series Near a Singular Point of the Circle of Convergence, by Mr. G. H. Hardy; The Intersection of Two Conic Sections, by Mr. J. A. H. Johnston.

In *Science* of August 18 Professor G. A. Miller states that at the celebration of the last birthday of the Emperor of Germany Professor Harzer delivered a long address on the 'Exact Sciences in Old Japan.' Although Professor Harzer is an astronomer, he devoted nearly his entire address to the history of mathematics, saying that the 2,000 mathematical works in the royal library of Tokio, some of which date back to 1595, are a sufficient guarantee of high esteem for mathematical knowledge. As the Japanese mind is very practical, it is to be expected that their mathematical achievements are in very close touch with practical problems and are foreign to those fields of mathematics which border on philosophy. The determination of the area of the circle in terms of its diameter is one of the most important of these practical problems and the Japanese have taken especial interest in developments which are useful to obtain an approximate solution of this problem.

The most surprising fact about Japanese mathematics is that, formerly, while the most elementary parts were regarded as common property, the more advanced results were regarded as secrets which should be communicated to a very few. In fact, an oath of secrecy was required of those who wished to hear lectures on advanced mathematics. European history furnishes a parallel to this in the Pythagorean school, but it is so totally different from the modern spirit that its existence 2,000 years after Pythagoras was unexpected. Fortunately all this has been recently changed to such an extent that a history of Japanese mathematics could be published a few years ago. This is by Tsuruichi Hayashi and a chapter of it entitled "A Brief History of Mathematics in Japan," has been translated into English and published in *Nieuw Archief Voor Wiskunde*, 1904, pp. 296-324, and 1905, pp. 325-361.

We learn from *Science* that the conference held at Asbury Park on July 5, 1905, for the purpose of discussing the advisability of organizing a national society of teachers of mathematics, was attended by a large delegation. After some discussion a resolution was adopted to the effect that a national society of teachers of mathematics and science be organized, and the details of the organization were referred to the following executive committee: Professor Thomas S. Fiske (chairman), New York; Professor C. E. Comstock, Peoria, Ill.; Professor E. R. Hedrick, Columbia, Mo.; Mr. Franklin T. Jones, Cleveland, O.; Professor William H. Metzler, Syracuse, N. Y.; Mr. Edgar H. Nichols, Cambridge, Mass.

A report of the proceedings of this committee will be published in *School Science and Mathematics*.

Following is a list of the associations represented at the Asbury convention, with names of delegates representing each association.

New England Mathematics Teachers Association.—Chas. E. Bouton, Harvard University; Paul Capron; Mr. Nichols, Brown and Nichols School, Cambridge.

Association of Teachers of Mathematics in the Middle States and Maryland.—John C. Bechtel; Fletcher Durell, Lawrenceville, N. J.; A. Newton Ebaugh; Miss Susan C. Lodge; Donald C. MacLaren; Wm. H. Metzler, Syracuse Univer-

sity; J. T. Rorer, Central High School, Philadelphia; Arthur Schultze, High School of Commerce, New York; H. C. Whitaker.

Central Association of Science and Mathematics Teachers.—Otis W. Caldwell; Jos. V. Collins; C. E. Comstock; G. W. Greenwood; Charles H. Smith; Charles M. Turton; J. W. Young, Charles W. Wright.

Missouri Society of Teachers of Mathematics.—F. T. Appleby; J. S. Bryan, Central High School, St. Louis; H. Clay Harvey; E. R. Hedrick; B. F. Johnston; John R. Kirk; J. W. Whiteye.

Chicago and Cook County High School Teachers' Association.—Edward E. Hill; Fred R. Nichols; Charles M. Turton.

Mathematical Section of Michigan School-Masters' Club.—Miss Emma C. Ackermann.

New York State Science Association, Mathematical Department.—Glenn M. Lee.

North Eastern Ohio Center, G. A. S. and M. T.—Lemar T. Beman, Cleveland High School; Charles A. Marple.

Ohio Association of Teachers of Mathematics and Science.—Franklin T. Jones; Wm. McLair.

St. Louis Association of Science and Mathematics Teachers.—Wm. Schuyler, McKinley High School, St. Louis.

BOOKS.

A College Algebra. By Henry Burchard Fine, Professor of Mathematics in Princeton University. viii+595 pages, Cloth. Boston, New York, Chicago, and London: Ginn & Co.

Those who have read Professor Fine's *The Number System of Algebra* will expectantly turn, in this volume, to part first, entitled, Numbers. Their anticipations will be fully met. Under the influence of the Cantor-Dedekind school of thought recent years have shown remarkable advancement of the theory of irrationals and of cardinal number. The elements of this theory lie at the basis of algebra and we have here, we believe, for the first time, a teachable treatise based on these fundamental concepts.

The treatment of the usual subject-matter of college algebra is also admirable. Special features of the text are, short chapters on Interpolation, on Variation, and on Probability, and a moderately long treatment of the elements of the theory of equations. The work ends with a rigid proof of the existence theorem of algebra.

The author has given little space to special devices but seeks "to assist the student to really master the general methods of the science," and the logical interrelations which connect its parts. The work is in every sense a classic. G.

Elements of Descriptive Geometry. By Chas. E. Ferris, Professor of Mechanical Engineering in the University of Tennessee. Cloth, 8vo, 135 pages. New York, Cincinnati, Chicago: American Book Co.

The author follows the custom of most draftsmen and does all his work in the third

quadrant. The exercises are well graded and the illustrative drawings are plentiful and well executed. The first six chapters treat the usual subject matter of the subject. Chapter VII is on Warped Surfaces; Chapter VIII treats the subject of Shades and Shadows, and Chapter IX is devoted to a brief treatment of Perspective. G.

The Elements of Geometry. By Walter N. Bush and John B. Clarke, of the Polytechnic High School of San Francisco. 355 pages. Price, $1.25. Chicago, Boston, New York: Silver, Burdett & Co.

On page 255 of his book entitled "The Teaching of Elementary Mathematics" Professor David Eugene Smith of Columbia University gives the following criteria for a usable text-book in Geometry, viz:

1st. A sequence of propositions which is not only logical, but psychological;

2nd. Exact statements;

3rd. Proofs which are models of excellence for the pupil to pattern after;

4th. Abundant exercises from the beginning with practical suggestions to methods of attacking them.

Judged by this standard "The Elements of Geometry" by Bush and Clarke should prove a very successful book. The sequence of propositions is a logical one; cognate theorems are put in special groups or classes. The merit of the arrangement from the psychological point of view has been tested by the authors during the many years they have devoted to the preparation of the work. The statements and definitions are accurate and clear. In the proofs the hypothesis and conclusion are carefully separated, thus not only making the path easier for the student, but also giving him a standard for the presentation of his solution of "originals." The exercises are both numerous and well selected.

The authors are to be commended for arranging the proofs so that it should not be necessary to turn the page and then have to turn back for the figure. As in all modern texts, the diagrams are carefully prepared and are very good. An excellent index at the end enables one to find easily any desired place. ALMA E. KLUNDER.

Biology and Mathematics. By George Bruce Halsted. 15 pages. An address delivered before Section A of the American Association for the Advancement of Science, Philadelphia, December 29, 1904. Reprinted from *Science*, N. S., Vol. XXII, No. 554, pages 161-167, August 11, 1905.

Professor Halsted discusses with great force and insight the analogies between the Darwinian theory of variation and the continuous variable, on the one hand, and the more recent theory of mutation and discontinuous variation, on the other. F.

ERRATA.

Equation (3) on page 122 should read,

$$x=\frac{p}{3c}\left[\frac{F'(0)+F'(p)}{2}+F'\left(\frac{p}{3}\right)+F'\left(\frac{2p}{3}\right)\right]-\frac{p^2}{108c}\left[F''(p)-F''(0)\right] \ldots\ldots (3).$$

Line 10 from bottom, on page 134, should read,

$$\begin{aligned}2S&=\frac{1}{2.3.4}+\frac{1}{5.6.7}+\frac{1}{8.9.10}+\ldots\ldots\\&=\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}\\&\quad+\tfrac{1}{3}-\tfrac{2}{4}+\tfrac{1}{5}\\&\qquad+\tfrac{1}{4}-\tfrac{2}{5}+\tfrac{1}{6}\\&\qquad\quad\vdots\quad\vdots\quad\vdots\\&=\tfrac{1}{2}-\tfrac{1}{3}=\tfrac{1}{6}.\end{aligned}$$

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SOME NEW METRICAL PROPERTIES OF CONIC CURVES.

By REV. ALAN S. HAWKESWORTH.

Theorem 1. If through the extremities of a double ordinate to any diameter of an ellipse right lines be drawn from the extremities of said diameter; then such lines will ever intersect upon a hyperbola, having said diameter and its conjugate in common with the ellipse; and thus asymptotes which are both parallel to and bisected by the supplementary chords uniting said conjugate diameters. While lastly, the double ordinate of the said point on the hyperbola, parallel to the given double ordinate in the ellipse, will cut, with it, their common bisecting diameter harmonically.

And conversely, of course, lines through the extremities of any double ordinate in a hyperbola from the ends of its diameter, will meet upon an ellipse having said diameter and its conjugate in common; and the ordinates of the corresponding points will cut their common diameter harmonically.

For let pTq be a double ordinate in an ellipse [Fig. 1], cutting its bisecting diameter DCD' in T; and let $D'pP$ and qDP meet in P. Draw PNQ parallel to pTq, meeting DCD' in N, and pD in Q. And draw ECE' the conjugate diameter to DCD'.

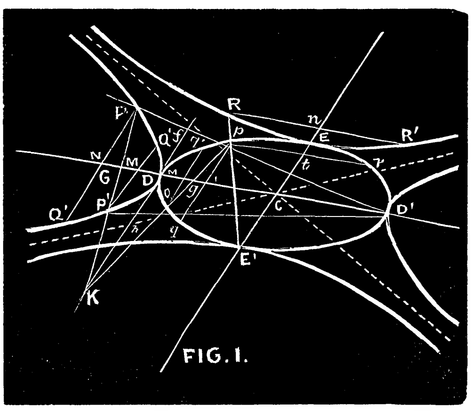


FIG. 1.

Then qTD and PND being similar triangles; and likewise pTD' and PND'; while $qT=pT$, and $PN=QN$; therefore $pT : DT=qT : DT=PN : DN$, while $pT : D'T=PN : D'N$. So that $pT^2 : DT.D'T=PN^2 : DN.D'N=QN^2 : DN.D'N=CE^2=CD^2$; which latter ratio belongs to both ellipse and hyperbola. And thus P and Q must ever lie on a hyperbola, having DCD' and ECE' for its conjugate diameters also.

In a precisely similar way, were ptr, parallel to DCD', the double ordinate taken; and lines $E'pR$ and rER drawn through p and r from the ends of its bisecting diameter ECE'; then such lines $E'pR$ and rER can be shown to meet upon a hyperbola, having ECE' and DCD' as its conjugate diameters also; and which is thus the curve conjugate to the first hyperbola.

And conversely, of course; were the double ordinate PNQ, or RnR' taken in the hyperbola, lines PpD' and QDp, or RpE' and $R'Ep$, can be shown to meet upon point p on an ellipse, having DCD' and ECE' for its conjugate diameters.

And next; the conjugate diameters DCD' and ECE' being common to both ellipse and hyperbola, by a law of the latter curve, the supplementary chords ED' and $E'D'$ must be parallel to one asymptote, and be bisected by the other.

While lastly, the parallel double ordinates PNQ and pTq [or RnR' and ptr] divide their common bisecting diameter DCD' [or ECE'] harmonically. For $DT{:}DN{=}qT{:}PN{=}pT{:}PN{=}D'T{:}D'N$. And similarly $Et{:}En{=}E't{:}E'n$.

Scholium. But note. In all the foregoing we have considered, in the hyperbola, merely those double ordinates that cut *one* branch. Those which cut *both* branches will be dealt with later, in Theorem 5. From which it will be evident that the true analogue, in the hyperbola, to the minor ordinates of the ellipse, is not its own minor ordinate, cutting both branches; but is rather the major ordinate of its conjugate.

Corollary 1. The four tangents at $DD'E$ and E' are common to ellipse and hyperbola.

Corollary 2. PT and QT will be the tangents to the hyperbola from T; and Rt, $R't$ the tangents from t to its conjugate. While in like manner pN, qN will be the tangents to the ellipse from N; and pn, rn its tangents from n.

Corollary 3. The parallel double ordinates pTq and PNQ, or ptr and RnR', through any two such corresponding points pP, or pR, must therefore cut the curves again in two fresh corresponding points qQ, or rR'. So that the determining lines to such double ordinates can be drawn through either extremity of its diameter indifferently. As *e. g.* $D'pP$ and $D'qQ$; or pDQ and qDP; $E'pR$ and $E'rR'$; or rER and pER'.

Corollary 4. When the axi of an ellipse are taken as our conjugate diameters, then the resultant hyperbola will have its axi either coincident, or reversed, according as the double ordinate in the ellipse was major or minor.

[NOTE. As first drafted, the foregoing and following theorems and corollaries considered merely this special case. But acting upon a hint from Professor Dickson, that by Projectional Geometry, Theorem 1 was true for *any* diameter, all were readily thrown into their present general form, by merely reading conjugate diameters for axi.]

Corallary 5. When the common conjugate diameters are equal, then evidently the projected hyperbola is equilateral, with the produced axi of the ellipse for its asymptotes; and conversely.

Corollary 6. While if a circle be taken as our ellipse, the hyperbola is again an equilateral.

Corollary 7. But on the other hand, when the ellipse passes into a parabola, then the projected conic has become a second and equal parabola, reversed in direction.

For obviously, if pTq be a double ordinate of a parabola [Fig. 2] and CTD its bisecting diameter, then since $pT=qT$, $pP=2DT=qQ=TN$; so that $pPQq$ is a parallelogram, for any position of the double ordinate pTq. Hence PQ lie on a parabola, equal to pDq, with $CTDNC'$ as a common diameter; and fDh as a common tangent. While DT ever equals DN; and pN, qN are the tangents from N to pDq; even as PT, QT are those from T to PDQ. While PQ lie on, and are definable by lines pDQ and qDP; as well as on lines pP and qQ, parallel to CDC'.

Theorem 2. If then pP and $p'P'$, or pP and $p'Q'$, be two pairs of corresponding points, either upon an ellipse and its projected hyperbola [Fig. 1], or upon a pair of equal parabolas [Fig. 2], said points being determined, either by the double ordinates pTq and PNQ, $q'mp'$ and $Q'MP'$, through the harmonic points TN and mM upon diameter DCD'; or else given by the intersection of the two curves by the lines $D'pP$ and $D'p'P'$ [or pDQ and $p'DQ$] through a common extremity D' [or D] of said diameter DCD'. Then the resultant corresponding chords pp' and PP' [or pp' and QQ'] both cut the diameter in two fresh harmonic points g and G; and also meet upon the tangent DK [or $D'K'$] of that extremity of the diameter through which the determining lines were not drawn.

For let chords pp' and PP' be taken in the ellipse and hyperbola [Fig. 1] determined by lines $D'pP$ and $D'p'P'$. Draw the double ordinates pTq, $p'mq'$, $P'MQ'$ and PNQ; and let the tangents at D cut $D'pfP$ in f; and $D'p'h'P'$ in h. Then T and N must be harmonic points [Theorem 1, Corollary 3]; and likewise m and M. Therefore $fp:fP=DT:DN=D'T:D'N=D'p:D'P$; and $hp':hP'=Dm:DM=D'm:D'M=D'p':D'P'$. So that $D'pfP$ and $D'p'hP'$ are harmonic ranges, whose pencil rays pp', fh, PP' must concur in K a summit of their quadrilateral; the other two summits being D', and a second point upon the tangent $fDhK$.

And in like manner, were chords pp' and QQ' chosen, they can be shown to concur upon the tangent at D'; the other two summits of their quadrilateral being D, and a second point upon the tangent at D'. So that in both cases $Dg:D'g=DG:D'G$.

While if chords pq' and PQ' [or QP'], determined by lines $D'pP$ and $D'q'Q'$ [or pDQ and $q'DP'$] be the ones taken; these can again be shown to meet upon the tangent at D [or D']; and to cut harmonically DCD' in G and g. And all the foregoing is, of course, equally true for any corresponding chords in the ellipse and conjugate hyperbola, that cut diameter ECE'.

Lastly; in the pair of equal but opposite parabolas [Fig. 2], which have diameter CDC' and its tangent fDh in common, let the chords pp' and PP' be taken; determined by lines pP and $p'P'$ parallel to CDC'. Draw the double ordinates pTq, $p'mq'$, $P'MQ'$, and PNQ; let the tangent of D meet pfP in f and $p'hP'$ in h; and let pp' and PP' cut diameter CDC' in g and G, respectively. Then since $pf=fP$, and $p'h=h'P$, pp' and PP' must concur in K on $fDhK$; while $gD=DG$.

Were pp' and QQ' the chosen chords [Fig. 2], determined by lines pDQ and $p'DQ'$; then since $pD=DQ$ and $p'D=DQ'$; pp' and QQ' must be parallel; and again $gD=DG$. While if we chose chords pq' and PQ', they must again concur on the tangent of D; and cut diameter CDC' at equal distances from D.

Corollary 1. If, therefore, any double ordinate in the ellipse, say pr, be considered with relation to its parallel diameter [*e. g.* DCD']; since they cut at infinity, the corresponding chord in the hyperbola, determined by lines from an extremity of the same diameter [*i. e.* $D'p$ and $D'r$; or Dp and Dr], must pass through the common center of the curves. And conversely, of course, if a hyperbola's double ordinate is considered in relation to its parallel diameter, then its corresponding chord in the ellipse passes through the common center C.

Theorem 3. But if in place of both determining lines running through the same extremity of a diameter, we have one through each extremity—as *e. g.* Fig. 3, $D'pP$ and DqQ,—then the connecting chords pq and PQ will cut the diameter in the *same* point; which will be internal or external, according as the two pairs of chosen points fell upon the opposite, or the same side of said diameter.

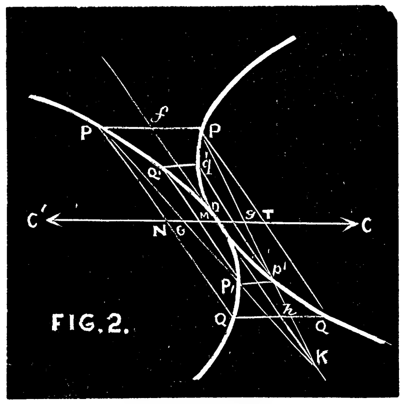


FIG. 2.

For, let pq and PQ be the chosen chords [Fig. 3], in ellipse and hyperbola, connecting the pairs pP and qQ determined by $D'pP$ and DqQ. And let the tangent at D cut $D'pP$ in f; while that at D' cuts DqQ in h. Draw PN, pT, qt, Qn, the ordinates of points Ppq and Q to diameter $D'CD$.

Then, as before, $fp:fP=DT:DN=D'T:D'N=D'p:D'P$; while $hq:hQ=D't:D'n=Dt:Dn=Dq:DQ$. So that $D'pfP$, and $DqhQ$ are harmonic ranges; and thus, if DD', pq, fh, and PQ be joined, they must concur in G upon DD'.

And similarly; if $p'P'$ and qQ be chosen on the same side of DCD', their chords $p'q$ and $P'Q$, in like manner, can be shown to concur in H on DCD' produced. Or if points on the ellipse and the conjugate hyperbola be chosen, determined by lines through E and E', their chords will concur on diameter ECE'.

While in the pair of parabolas [Fig. 2], if chords pq' and PP' be chosen, determined by lines pfP and $q'DP'$; then again, since $pf=fP$; and $q'D=DP'$, the pencil rays pq' and PP' must concur in G upon CDC'; and fG be parallel to $q'DP'$, since they meet upon the ideal tangent at D' at infinity. Likewise PQ' and pp' concur in g; and fg is parallel to $Q'Dp'$.

Theorem 4. And hence, since the tangents of, or chords through corresponding points upon an ellipse and its auxiliary circle, determined by their common major ordinates, concur on the major axis. And similarly, those through the corresponding points upon the ellipse, and a circle whose diameter is the minor axis, determined by their common minor ordinates, concur on the minor axis.

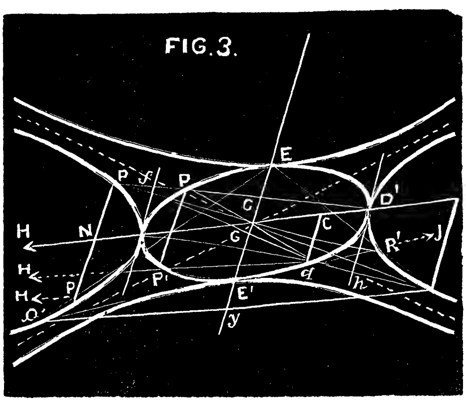
So in like manner, if we draw the tangents of, or chords through the corresponding points upon a hyperbola and its auxiliary circle; or upon the conjugate and its auxiliary circle; said corresponding points being determined by the polar ordinates to the respective major axis; then the said corresponding tangents or chords will cut that axis harmonically [Theorem 2].

Except where the chord of the hyperbola cuts both branches, and each extremity is also upon the same side of the respective major axis, as the corresponding extremity of the circle's chord. In which special case the said corresponding chords will cut the axis in the *same* point [Theorem 3].

[*Scholium.* And note here again, that the hyperbola which bears the same relation to the circle on the minor axis as the ellipse does to its minor auxiliary circle, is the *conjugate* hyperbola.]

Theorem 5. If from the extremities of a double ordinate in a hyperbola, which cuts both branches of the curve, right lines be drawn through the extremities of its bisecting diameter, these will intersect upon a second point on the curve; the double ordinate through which will cut said diameter harmonically to the first.

Let QyQ' be a double ordinate [Fig. 3], cutting both branches; with ECE' as its bisecting diameter; and let QRE and $Q'E'R$ intersect in R. Draw RvP' parallel to QyQ', and meeting ECE' in v.



Then by similar triangles, $Qy:yE' = Q'y:yE' = Rv:vE'$; and $Qy:yE = Rv:vE$. So that both $yE':yE = vE':vE$, and also $Qy^2:yE'.yE = Rv^2:vE'.vE$; or $Qy^2:Rv^2 = Cy^2 - CE'^2 : CE'^2 - Cv^2$ [Euclid II., 5 and 6].

But since yv divide EE' harmonically, $Cy:CE' = CE':Cv$. For $yE':yE = vE':vE$; or $Cy - CE' : Cy + CE' = CE' - Cv : CE' + Cv$; so that $Cy:CE' = CE':Cv$ [Euclid V., E]. And hence $Cy^2:CE'^2 = CE'^2:Cv^2 = Cy^2 + CE'^2 : CE'^2 + Cv^2 = Cy^2 - CE'^2 : CE'^2 - Cv^2$. And thus $Qy^2:Rv^2 = Cy^2 + CE'^2 : CE'^2 + Cv^2$; or $Qy^2 : Cy^2 + CE'^2 = Rv^2 : CE'^2 + Cv^2$.

Now by a well known theorem, $D'n.Dn : Qn^2 = CD'^2 : CE'^2 = Cn^2 - CD'^2 : Qn^2 = Qy^2 - CD'^2 : Cy^2 = Qy^2 : Cy^2 + CE'^2$; which as we have seen $= Rv^2 : CE'^2 + Cv^2$. And thus Rv is also an ordinate to diameter ECE'; and hence R is a point upon the curve.

And conversely, of course, if RvP' were the double ordinate chosen, cutting diameter ECE' internally, then ERQ and $P'E'Q$ can be shown to meet in Q upon the curve, whose ordinate Qy will cut ECE' externally, and harmonically to v.

Corollary 1. If the tangent at E' cuts QRE in say w; then $Qw:QE = yE' : yE = vE':vE = Rw:RE$. So that $QwRE$ will ever be a harmonic range.

Corollary 2. And hence when Q is taken at infinity, and thus QE becomes parallel to its asymptote; R must coincide with D', and $D'w=D'E$.

Corollary 3. While on the other hand, when Q coincides with the intersection of the curve by the tangent at E', then QRE must cut the curve infinitesimally close to Q; or in other words, has become the tangent at Q. A result also directly evident from the general law that if Pt be the tangent, and Pn the normal to any diameter ECE'; then $Ct.Cn=CE^2$.

Corollary 4. Therefore E [or E'] is a pole, and the tangent at E' [or E] its polar, for the hyperbola $QRD'PDQ'$. And similarly, D [or D'] is a pole, and the tangent at D' [or D] its polar, for the conjugate curve.

Corollary 5. And thus if any line through E [or E'] cut the curve in RQ, then $E'R$ and $E'Q$ [or ER and EQ] will cut the conjugate diameter DCD' in say JH at equal distances from C. For $RE'Q'$ being collinear, $CH:CJ=Qy:Q'y$. And thus if EJ be joined, CEJ and $CE'H$ will be equal triangles, and EJ be parallel to $QE'H$; and similarly, EH parallel to $E'RJ$. So that if L be the point where ERQ cuts DCD', then ELJ and QLH are similar triangles; and $EL:LQ=JL:LH$.

Theorem 6. If pp' be any chord in a conic curve, DCD' its bisecting diameter, and q any other point on the curve; then chords pq and $p'q$ will ever cut DCD' harmonically. Except when the chosen chord pp' is one in a hyperbola, which cuts both branches. In which case one of the said "harmonic segments" of the bisecting diameter will be reversed in direction.

Let pp' be a chord in the ellipse [Fig. 3], DCD' its bisecting diameter, and q any other point on the curve. Let pq and $p'q$ cut DCD' in G and H, respectively. Let PP' be the chord and Q the point in the projected hyperbola, corresponding to pp' and q; and defined by lines pDP', $p'DP$, and DqQ. Then chord PP' must be also definable by lines $D'pP$ and $D'p'P'$ [Theorem 1, Corollary 3]. And thus while chords pq and $P'Q$ cut DCD' harmonically in G and H [Theorem 2]; chords pq and PQ, on the other hand, must cut it in the same point G [Theorem 3]; while $p'q$ and $P'Q$ cut it in H. So that chords pq and $p'q$ in the ellipse cut DCD' harmonically; and likewise chords PQ and $P'Q$ in the hyperbola.

In the same way the lines joining any point to the extremities of any chord cutting but one branch in the conjugate hyperbola cut the bisecting diameter harmonically. While in the twin parabolas [Fig. 2], if pq be the chord, bisected by CDC', and p' be the point; pp' and $p'q$ will cut CDC' harmonically. For let PQ be the chord and P' the point in the second parabola, defined by lines pP, $p'P'$, and qQ, parallel to CDC'. Then PQ also lie upon lines qDP and pDQ [Theorem 1, Corollary 7]. So that chords qp' and QP' will cut CDC' harmonically in Gg [Theorem 2]; while pp' and QP' will cut it in the same point g [Theorem 3], and thus pp' and qp' cut CDC' harmonically.

Lastly; let the chosen chord in the hyperbola be one QyQ' cutting both branches of the curve [Fig. 3], with ECE' as its bisecting diameter, and P as the point. Then chords PQ and PQ' will cut ECE' in say z and z', respectively; coincident with its intersection by the ordinate and tangent of some fourth point

on the curve. So that $Cz.Cz'=CE^2$; and if either Cz or Cz' be reversed in direction, ECE' will be cut harmonically.

For if R or P' were taken as our fixed point; then as shown in Theorem 5, RQ and RQ', or $P'Q'$ and $P'Q$ will cut ECE' in E and E'. While if a point on the curve—say Q''—cut by the chord through Q or Q', parallel to ECE', were chosen; then obviously, chord $Q''Q'$, or $Q''Q$ must pass through C. While lastly, if the fixed point be taken infinitesimally close to either Q or Q', then its lines through Q and Q' have become the tangent and double ordinate of said points; and thus the product of their segments again equals CE^2.

But since this is now true for *six* points on the curve, by the principle of continuity it must be true for *any* point on said curve. And hence PQ and PQ' must cut ECE' in say z and z'; so that $Cz.Cz'=CE^2$; the tangent to the ordinate of z to ECE' passing through z'; and conversely; and thus if either Cz or Cz' be reversed in direction, ECE' will be cut harmonically.

Corollary 1. Therefore in the ellipse chords pB and pB', through the extremities of the minor axis BB', will ever cut AA', the major axis, harmonically. And similarly, pA, pA' cut BB' harmonically.

Corollary 2. While in the hyperbola, since lines through P parallel to the asymptotes will represent both the lines to the *ideal* minor axis, and those to the *ideal* diameter conjugate to DCD', both at infinity. Therefore such lines will cut both DCD' and the major axis AA' to the curve harmonically; and will intersect both ECE', and the accepted minor axis BB'—the major of the conjugate—in the same distances from C, though one of them reversed, as are a harmonic pole and polar.

And in like manner PA, PA' will ever cut BB' in its intersection by the ordinate and tangent of the same fourth point on the curve; even as PD, PD' also cut ECE'.

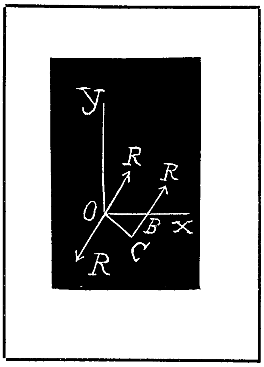
TO FIND THE EQUATION TO THE STRAIGHT LINE WHICH IS THE DIRECTION OF THE RESULTANT OF A SYSTEM OF FORCES ACTING IN ONE PLANE.

By G. B. M. ZERR, A. M., Ph. D.

This article is not presented with the idea of developing something new, but for the purpose of collecting the old together into a form convenient for the solution of such problems as the example given below.

All the forces in one plane can be resolved into the resultant R acting through the origin O and a couple H.

Let G be the moment of the couple, θ the angle R makes with the axis of abscissas, P any one of the forces, β the angle P makes with the axis of abscissas. Also, let $X=P\cos\beta$, $Y=P\sin\beta$.



Then $G=\Sigma(Yx-Xy)$, $\tan\theta=\Sigma Y/\Sigma X$. Transform G into a couple having its two forces each $=R$: the one acting at O directly opposite to R, and, hence, destroying it; the other, at a distance $CO=G/R$ from the origin.

Then $OB=OC\operatorname{cosec}\theta=\dfrac{G}{R}\cdot\dfrac{R}{\Sigma Y}=\dfrac{G}{\Sigma Y}$.

The equation to the line required is, therefore, $y=\tan\theta(x-OB)=(\Sigma Y/\Sigma X)(x-G/\Sigma Y)$.

$\therefore y\Sigma X=x\Sigma Y-G$ .... (1).

Problem. Each element of the arc of an elliptic quadrant is acted on by a force in the normal proportional to the ordinate of that point. Find the equation to the straight line along which the resultant acts.

Let (m, n) be the coördinates of the point.

Then $y-n=\dfrac{a^2n}{b^2m}(x-m)$ is the normal.

$\therefore \tan\beta=\dfrac{a^2n}{b^2m}$, $P=Cn$, suppose.

$$\therefore Cn\cos\beta=\frac{Cb^2mn}{\sqrt{(b^4m^2+a^4n^2)}}=\frac{Cb^2mn}{a\sqrt{[b^4+(a^2-b^2)n^2]}}=X,$$

$$Cn\sin\beta=\frac{Ca^2n^2}{\sqrt{(b^4m^2+a^4n^2)}}=\frac{Ca^2n^2}{a\sqrt{[b^4+(a^2-b^2)n^2]}}=Y,$$

$$\frac{C(a^2-b^2)n^2m}{a\sqrt{[b^4+(a^2-b^2)n^2]}}=Yn-Xm,\quad \Sigma X=\int_0^b Xds\Big/\int_0^b dn=\frac{1}{b}\int_0^b Xds.$$

Similarly, $\Sigma Y=\frac{1}{b}\int_0^b Yds,\quad G=\Sigma(Yx-Xy)=\frac{1}{b}\int_0^b(Yx-Xy)ds,$

$$ds=\frac{\sqrt{[b^4+ca^2-b^2)n^2]dn}}{b\sqrt{(b^2-n^2)}},\quad m=\frac{a}{b}\sqrt{(b^2-n^2)}.$$

$$\therefore \Sigma X=\frac{C}{b}\int_0^b ndn=\tfrac{1}{2}bC,\quad \Sigma Y=\frac{Ca}{b^2}\int_0^b\frac{n^2dn}{\sqrt{(b-n^2)}}=\tfrac{1}{4}\pi aC,$$

$$G=\frac{C(a^2-b^2)}{b^3}\int_0^b n^2dn=\frac{C}{3}(a^2-b^2).$$

Substituting in (1) we get, $\tfrac{1}{2}bCy=\tfrac{1}{4}\pi axC-\frac{C}{3}(a^2-b^2).$

$\therefore 6by-3\pi ax+4a^2-4b^2=0.$

We generalize as follows: Suppose each element of the arc of the curve $y=f(x)$, between the limits $y=0$ and $y=k$, $(x=h)$, is acted on by a force in the normal proportional to the ordinate of the point; to find the equation to the line of action of the resultant.

Let (m, n) be the point, and let $\sqrt{\{1+[f'(m)]^2\}}=\mathrm{D}$.

Then $\tan\beta=-\frac{dm}{dn}=-\frac{1}{f'(m)}$.

$$\therefore X=Cn\cos\beta=\frac{Cnf'(m)}{\mathrm{D}},\quad Y=Cn\sin\beta=-\frac{Cn}{\mathrm{D}},$$

$$(Ym-Xn)=-\frac{Cn[m+nf'(m)]}{\mathrm{D}},\quad ds=\frac{\mathrm{D}dn}{f'(m)}=\mathrm{D}dm.$$

$$\therefore \Sigma X=\frac{C}{k}\int_0^k ndn=\tfrac{1}{2}Ck,\quad \Sigma Y=-\frac{C}{k}\int_0^k\frac{ndn}{f'(m)},\quad G=\frac{C}{k}\int_0^k\frac{n[m+nf'(m)]dn}{f'(m)}.$$

$$\therefore G=\frac{C}{k}\int_0^k\frac{nm\,dn}{f'(m)}+\tfrac{1}{3}Ck^2.$$

$$\therefore 3ky+\frac{6x}{k}\int_0^k\frac{ndn}{f'(m)}=\frac{6}{k}\int_0^k\frac{nm\,dn}{f'(m)}+2k^2,$$

is the equation to the required line.

Consider the special example, $f(x)=2\sqrt{ax}$, whence $y^2=4ax$. Let the arc extend from the vertex to one extremity of the latus-rectum.

$\therefore k=2a,\quad m=n^2/4a,\quad f'(m)=\sqrt{(a/m)}=2a/n.$

$$\therefore \frac{6}{k}\int_0^k\frac{ndn}{f'(m)}=\frac{6}{2a}\int_0^{2a}\frac{n^2dn}{2a}=4a,\quad \frac{6}{k}\int_0^k\frac{nm\,dn}{f'(m)}=\frac{6}{2a}\int_0^{2a}\frac{n^4\,dn}{8a^2}=\frac{12a^2}{5}.$$

$\therefore 6ay+4ax=\tfrac{12}{5}a^2+8a^2=\tfrac{52}{5}a^2.$

$\therefore 15y+10x=26a$, is the required equation.

DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

SOLUTIONS OF PROBLEMS.

NOTE. The following problems were received too late for publication: Calculus No. 200, and Geometry No. 263, solved by G. W. Greenwood. Credit is also given to J. Scheffer for solutions of Calculus No. 201, Diophantine Analysis No. 127, and Geometry No. 262.

ALGEBRA.

237. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $x^2+y+z=12$.....(1) ; $x+y^2+z=8$.....(2) ; $x+y+z^2=6$.....(3).

Solution by L. E. NEWCOMB, Los Gatos, Cal.

Since (3) from (1) gives $x^2-x=6+z^2-z$, $\therefore$ $x=\frac{1}{2}\pm\sqrt{(6\frac{1}{4}+z^2-z)}$.....(4); (3) from (2) gives $y=\frac{1}{2}\pm\sqrt{(2\frac{1}{4}+z^2-z)}$..........(5).

Substitute these values for x and y in (3); then $\frac{1}{2}+\sqrt{(6\frac{1}{4}+z^2-z)}+\frac{1}{2}+\sqrt{(2\frac{1}{4}+z^2-z)}+z^2=6$.

Whence $[\sqrt{(6\frac{1}{4}+z^2-z)}+\sqrt{(2\frac{1}{4}+z^2-z)}]^2=(5-z^2)^2$.........(6).

After expansion and transposition, (6) becomes $2\sqrt{[(6\frac{1}{4}+z^2-z)(2\frac{1}{4}+z^2-z)]}=z^4-12z^2+2z+16\frac{1}{2}$. Square both numbers; then $z^8-24z^6+4z^5+173z^4-40z^3-430z^2+100z+216=0$. The roots are $z=+1$, $+.204923$, $+2.39427$, $+3.48865$, -3.806118, -2.36109, -2.09446, $-.67044$.

Similarly two equations, one involving x, the other, y, are derived; or the values of x, y may be found from (4), (5), respectively. These values are:

$x=3$, $+3.3983$, -2.59649, -3.3642, -4.45405, -3.2664, $+4.06808$, $+3.21476$;
$y=2$, -1.5976, $+2.86394$, -2.80636, -4.0324, $+3.69155$, -2.4548, $+2.33574$.

Also solved by A. H. Holmes, J. Scheffer, G. B. M. Zerr, and the Proposer.

238. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{1}{n+1}+\frac{1}{3(n+3)}+\frac{1}{5(n+5)}+\text{..........}=\frac{1}{2}\left[\frac{1}{(n-1)}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\text{.....}+\frac{1}{l(n-l)}\right]$, n being an even positive integer and $l=n-1$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Putting successively $m=2, 4, 6, 8,$ ....$-m$, we get

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\text{........ ad inf.}=\tfrac{1}{2}[(\tfrac{1}{1}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{5})+(\tfrac{1}{5}-\tfrac{1}{7})+\text{........}]=\tfrac{1}{2}(1)=\tfrac{1}{2}.$$

$$\frac{1}{1.5}+\frac{1}{3.7}+\frac{1}{5.9}+\cdots\cdots=\tfrac{1}{4}[(\tfrac{1}{1}-\tfrac{1}{5})+(\tfrac{1}{3}-\tfrac{1}{7})+(\tfrac{1}{5}-\tfrac{1}{9})+\cdots\cdots]=\tfrac{1}{4}(1+\tfrac{1}{3}).$$

$$\frac{1}{1.7}+\frac{1}{3.9}+\frac{1}{5.11}+\cdots\cdots=\tfrac{1}{6}[(\tfrac{1}{1}-\tfrac{1}{7})+(\tfrac{1}{3}-\tfrac{1}{9})+(\tfrac{1}{5}-\tfrac{1}{11})+\cdots\cdots]=\tfrac{1}{6}(1+\tfrac{1}{3}+\tfrac{1}{5}).$$

....

$$\frac{1}{1(2m+1)}+\frac{1}{3(2m+3)}+\frac{1}{5(2m+5)}+\cdots\cdots\text{ad inf.}=\frac{1}{2m}\left[\left(1-\frac{1}{2m+1}\right)\right.$$

$$\left.+\left(\tfrac{1}{3}-\frac{1}{2m+3}\right)+\left(\tfrac{1}{5}-\frac{1}{2m+5}\right)+\cdots\cdots\right]=\frac{1}{2m}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\cdots\cdots\frac{1}{2m-1}\right).$$

Beginning in the finite series in the second member of this equation with the last term, we have

$$\frac{1}{1(2m+1)}+\frac{1}{3(2m+3)}+\frac{1}{5(2m+5)}+\cdots\cdots\text{ad inf.}$$

$$=\frac{1}{2m}\left[\frac{1}{2m-1}+\frac{1}{2m-3}+\frac{1}{2m-5}+\cdots\cdots\frac{1}{2m-(2m-1)}\right].$$

Putting $2m=n$, we get

$$\frac{1}{n+1}+\frac{1}{3(n+3)}+\frac{1}{5(n+5)}+\cdots\cdots=\frac{1}{n}\left[\frac{1}{n-1}+\frac{1}{n-3}+\frac{1}{n-5}+\cdots\cdots\tfrac{1}{1}\right]\cdots\cdots(\text{I}).$$

$$=\frac{1}{n}\cdot\frac{1}{n-1}+\frac{1}{n}\cdot\frac{1}{n-3}+\frac{1}{n}\cdot\frac{1}{n-5}+\cdots\cdots+\frac{1}{n}\cdot\tfrac{1}{1}$$

$$=\left(\frac{1}{n-1}-\frac{1}{n}\right)+\tfrac{1}{3}\left(\frac{1}{n-3}-\frac{1}{n}\right)+\tfrac{1}{5}\left(\frac{1}{n-5}-\frac{1}{n}\right)+\cdots\cdots\frac{1}{n-1}\left(\tfrac{1}{1}-\frac{1}{n}\right)$$

$$\therefore\ \frac{1}{n+1}+\frac{1}{3(n+3)}+\frac{1}{5(n+3)}+\cdots\cdots$$

$$=\left[\frac{1}{n-1}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\cdots\cdots\frac{1}{(n-1)1}\right]-\frac{1}{n}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\frac{1}{n-1}\right)\Bigg]\cdots\cdots(\text{II}).$$

Adding (I) and (II), we have

$$2\left[\frac{1}{n+1}+\frac{1}{3(n+1)}+\frac{1}{5(n+5)}+\cdots\cdots\right]=\frac{1}{n-1}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\cdots\cdots\frac{1}{(n-1)1}$$

$$\therefore\ \frac{1}{n+1}+\frac{1}{3(n+3)}+\frac{1}{5(n+5)}+\cdots\cdots$$

$$=\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\cdots\cdots\frac{1}{(n-1)1}\right]. \quad \text{Q. E. D.}$$

Also solved by G. W. Greenwood, G. B. M. Zerr, and the Proposer.

239. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $\sqrt[4]{(41+x)}+\sqrt[4]{(41-x)}=4$.

Solution by W. L. TRYON, Cornell University, Ithaca, New York.

Denote the equation by $m+n=4$.........(1). Then $m^4+n^4=82$..........(2).

Raising (1) to the fourth power, subtracting (2), and dividing by 2, we obtain

$$2mn(m+n)^2-m^2n^2=87,$$

$$i.\ e., \qquad m^2n^2-32mn=87.$$

$\therefore \quad mn=3$ or 29.

$m=1,\ 3,\ 2\pm5i$.

$\therefore \quad 41+x=1,\ 81,\ 41\pm840i$.

$x=\pm40,\ \pm840i$.

Also solved by P. S. Berg, G. W. Greenwood, J. J. Keyes, F. P. Matz, J. Scheffer, J. Edward Sanders, Jacob Westlund, and G. B. M. Zerr.

240. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve $a^2x+b^2y=ax^2+by^2=x^3+y^3$.

I. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

We have $(x^3+y^3)(a^2x+b^2y)=(ax^2+by^2)^2$, from which we obtain $xy=0$, and $ay=bx$. Substituting in either of the original equations we obtain for x, y the pairs of values $(0, 0)$, (a, b).

II. Solution by A. H. HOLMES, Brunswick, Maine.

From $a^2x+b^2y=ax^2+by^2$ we have $ax(a-x)=by(y-b)$........ (1).

From $ax^2+by^2=x^3+y^3$ we have $x^2(a-x)=y^2(y-b)$.........(2).

Dividing (1) by (2), $\dfrac{a}{x}=\dfrac{b}{y}$. $\quad\therefore y=\dfrac{b}{a}x$.

$$\therefore a^2x-ax^2=\frac{b^3x^2}{a^2}-\frac{b^3x}{a}. \qquad \therefore x=\frac{a^4+ab^3}{a^3+b^3};\ y=\frac{b}{a}x=\frac{a^3b+b^4}{a^3+b^3}.$$

Also solved by M. R. Beck, M. E. Graber, B. F. Finkel, J. E. Sanders, Elmer Schuyler, G. B. M. and the Proposer.

241. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Sum to infinity $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{144}+\frac{1}{840}+$..........(1).

Solution by ELMER SCHUYLER, Brooklyn, New York.

Separate the denominators into their prime factors; then (1) becomes

$$\frac{1}{1.2}+\frac{1}{1.3}+\frac{1}{1.2.4}+\frac{1}{1.2.3.5}+\frac{1}{1.2.3.4.6}\text{.........}(2).$$

Series (2) is equivalent to $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}\ldots\ldots\frac{n}{(n+1)!}\ldots\ldots$

*This becomes $\sum\limits_{n=1}^{\infty}\frac{n}{n+1!}=\sum\limits_{n=1}^{\infty}\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)=1.$

Also solved by G. B. M. Zerr, and the Proposer.

AVERAGE AND PROBABILITY.

165. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square parallel to one of the diagonals?

I. Solution by R. D. CARMICHAEL, Hartselle, Ala.

Their intersections along the other diagonal will be evenly distributed. The average length is thus readily seen to be one-half the diagonal$=\frac{1}{2}a\sqrt{2}$, where a=the side of the square.

II. Solution by J. EDWARD SANDERS, Hackney, Ohio.

Since the greatest length is $a\sqrt{2}$, and the least 0, the average length is $\frac{1}{2}(a\sqrt{2}+0)=\frac{1}{2}a\sqrt{2}$; or by calculus,

$$\triangle=\frac{1}{a}\int_0^a x\sqrt{2}\,dx=\frac{1}{2}a\sqrt{2}.$$

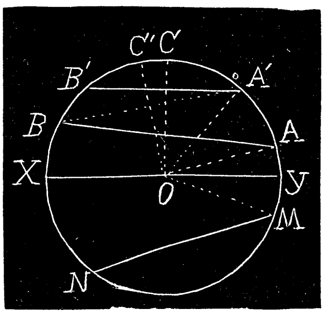
Also solved by F. P. Matz, and G. B. M. Zerr.

166. Proposed by F. P. MATZ, Sc. D., Ph. D.

Find the average area intercepted by two non-intersecting chords drawn at random in a given circle.

Solution by the PROPOSER.

First Case. The two chords AB and $A'B'$ may be on the *same* side of the diameter XY.



Let $OA=r$, $\angle OAB=\theta$, $\angle OA'B'=\phi$, $\angle YOA'=\psi$, and $\angle YOA=\omega$; then area

$AA'B'B=U_1=(\phi-\theta+\sin\phi\cos\phi-\sin\theta\cos\theta)r^2$.

$$\therefore A_1=\frac{\int_0^{\frac{1}{2}\pi}\int_\theta^{\frac{1}{2}\pi}\int_0^{2\pi}\int_\omega^{\frac{1}{2}\pi}U_1\,d\theta\,d\phi\,d\omega\,d\psi}{\int_0^{\frac{1}{2}\pi}\int_\theta^{\frac{1}{2}\pi}\int_0^{2\pi}\int_\omega^{\frac{1}{2}\pi}d\theta\,d\phi\,d\omega\,d\psi}\cdot$$

*C. Smith, *Treatise on Algebra*, p. 396, Ex. 2.

$$=\frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{8}\pi^2+\tfrac{1}{2}\theta^2-\tfrac{1}{2}\pi\theta+\tfrac{1}{2}\cos^2\theta-\tfrac{1}{2}\pi\sin\theta\cos\theta+\theta\sin\theta\cos\theta)d\theta=\tfrac{1}{6}\pi r^2.$$

Second Case. The two chords AB and $A'B'$ may be on *different* sides of the diameter XY; that is, the chord $A'B'$ now becomes the chord MN in position.

$\therefore$ Area $AMNB=U_2=(\phi+\theta+\sin\phi\cos\phi+\sin\theta\cos\theta)r^2$; and consequently,

$$A_2=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}U_2 d\theta\, d\phi\, d\omega\, d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta\, d\phi\, d\omega\, d\psi}$$

$$=\frac{4r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{8}\pi^2+\tfrac{1}{2}\pi\theta+\tfrac{1}{2}+\tfrac{1}{2}\pi\sin\theta\cos\theta)d\theta=\tfrac{1}{2}(\pi+1/\pi)r^2.$$

$\therefore A=\frac{1}{2}(A_1+A_2)=(\frac{1}{3}\pi+1/4\pi)r^2$, which is the required average area.

Corollary. If $r=1$, $A=\frac{113}{314}$; that is, the required average is slightly greater than one-third of the given circle.

CALCULUS.

201. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve $\int\int_0^{dy/dx}\frac{dw}{1+w^2}=0.$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$\int\int_0^{dy/dx}\frac{dw}{1+w^2}=0.\quad \int\left[\tan^{-1}w\right]_0^{dy/dx}=0.\quad \therefore\int\tan^{-1}\frac{dy}{dx}=0.\quad \therefore\frac{dy}{dx}=0.$

$y=C=$a constant.

202. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the complete primitive of $y=2px+ap^2$. Regard the primitive as the equation giving the arbitrary constant, and if the primitive has equal roots discuss the equation expressing that condition.

Solution by G. W. GREENWOOD, M. A.

Differentiating with regard to x we obtain an equation which may be written

$$p\frac{dx}{dp}+2x=-2ap.$$

Integrating, we have $3p^2x+2ap^3=c$, where c is a constant. Eliminating p between this equation and the original equation, we get

$$a^2c^2+2c(2x^3+3axy)-y^2(4ay+3x^2)=0.$$

This is the primitive equation. If the values of c are equal, then

$$(2x^3+3axy)^2+a^2y^2(4ay+3x^2)=0;\ i.\ e.,\ (x^2+ay)^3=0.$$

When the discriminating equation has a factor cubed, this factor equated to zero gives the cuspidal locus of the given family of curves. In fact, in the present case there is a cusp at the point $(\,a^{\frac{2}{3}}c^{\frac{1}{3}}\,,\ -\,a^{\frac{1}{3}}c^{\frac{2}{3}}\,)$.

GEOMETRY.

260. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Perpendiculars to the radius vector are drawn through points on $r=a+b\cos n\theta$. Find the radius of curvature of their envelope at a point at a given distance from the origin.

I. Solution by G. B. M. ZERR, A. M., Ph. D.

$r=a+b\cos n\theta=R\cos(\phi-\theta)=p$..........(1).

Differentiating (1) we get, $-bn\sin n\theta=R\sin(\phi-\theta)$.........(2).

$\therefore R^2=(a+b\cos n\theta)^2+b^2n^2\sin^2 n\theta$.........(3).

$\therefore R^2=p^2+b^2n^2-n^2(p-a)^2$.........(4).

(3) and (4) are both equations to the envelope.

Radius of curvature $=\rho=R(dR/dp)$.

$\therefore \rho=p+an^2-n^2p=r+an^2-n^2r=an^2+(1-n^2)(a+b\cos n\theta)$

$=a+b\cos n\theta-bn^2\cos n\theta=a-b(n^2-1)\cos n\theta$.

II. Solution by G. W. GREENWOOD. M. A.

Let $(p,\ \alpha)$ be the foot of the perpendicular from the origin upon the tangent at a point P of the envelope. Then at P,

$$\rho=p+d^2p/d\alpha^2=a+b\cos n\alpha-bn^2\cos n\alpha,$$

since $\qquad p=a+b\cos n\alpha.$

MECHANICS.

182. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

I have a tank, the lower part of which is a hemisphere 22 feet in diameter. The rest is a cylinder 22 feet in diameter, and altitude 28 feet. This tank is connected with the earth by a vertical stand-pipe 10 inches in diameter, 130 feet long, extending 2 feet into the tank. The tank is filled by a $2\frac{1}{2}$ inch pipe 65 feet long, having one right-angled elbow delivering the water into the bottom of the stand-pipe from a steam pump under 96 pounds gauge pressure. How long will it take to fill the pipe?

Solution by the PROPOSER.

Let h=effective head of water, W=weight of water discharged, l=65 feet =length of feed pipe, $d=2\frac{1}{2}$ inches$=\frac{5}{24}$ feet=diameter of feed pipe. Then hW=total energy consumed.

$\frac{Wv^2}{2g}$=kinetic energy giving motion to the water, $\frac{flv^2W}{2dg}$=energy of surface friction in pipe, $\frac{\beta v^2W}{2g}$=energy of resistance at entrance of pipe, $\frac{\delta v^2W}{2g}$=energy of resistance at elbow. $\therefore hW=\left(\frac{fl}{d}+\beta+\delta+1\right)\frac{Wv^2}{2g}$.

From many experiments, $f=.030268$, $\beta=.505$, $\delta=.9846$.

$\therefore h=(9.443616+.505+.9846+1)\frac{v^2}{2g}$; $\quad \frac{1}{2g}=.0155$.

$\therefore h=.184965v^2$ or $v=2.32517\sqrt{h}$.

$Q=\frac{\pi}{4}d^2v$=cubic feet delivered per second$=\frac{\pi\times\frac{25}{4}\times3600\times2.32517\sqrt{h}}{4\times144}$ $=285.342\sqrt{h}=C\sqrt{h}$ suppose=cubic feet per hour.

$C\sqrt{h}\,dt$=cubic feet delivered in time dt; πy^2dx=height dx of tank filled in time dt.

$\therefore C\sqrt{h}\,dt=\pi y^2dx$. $\quad\therefore t=\frac{\pi}{C}\int\frac{y^2dx}{\sqrt{h}}$.

Let $p=96$ pounds=gauge pressure of water on pump. Since one foot of water=.43302 pounds per square inch, p pounds=$2.30936p$ feet of water.

$\therefore h=2.30936p-130$. Let $B=2.30936p-128$; $B'=2.30936p-139$.

Let t=time of filling whole tank; t_1=time of filling the bottom two feet; t_2=time of filling the rest of the hemisphere; t_3=time of filling the cylinder.

The equation to the circle whose revolution about a diameter generates the sphere is $x^2+y^2=22x$.

$$\therefore t_1=\frac{\pi}{C\sqrt{h}}\int_0^2(22x-x^2)dx=\frac{124\pi}{3C\sqrt{h}};\quad t_2=\frac{\pi}{C}\int_2^{11}\frac{(22x-x^2)\,dx}{\sqrt{(B-x)}}.$$

$$\therefore t_2=\frac{\pi}{15\,C}\Big[(6x^2-220x+8Bx-440B+16B^2)\sqrt{(B-x)}\Big]_2^{11}$$

$$=\frac{\pi}{15C}\Big[(16B^2-352B-1694)\sqrt{(B-11)}-(16B^2-424B-416)\sqrt{(B-2}\Big]$$

$$t_3=\frac{121\pi}{C}\int_0^{28}\frac{dx}{\sqrt{(B'-x)}}=\frac{242\pi}{C}[\sqrt{(B')}-\sqrt{(B'-28)}].$$

$$t=t_1+t_2+t_3=\frac{\pi}{C}\Big[\frac{124}{3\sqrt{h}}+\tfrac{1}{15}(16B^2-352B-1694)\sqrt{(B-11)}$$

$$-\tfrac{1}{15}(16B^2-424B-416)\sqrt{(B-2)}+242[\sqrt{(B')}-\sqrt{(B'-28)}]\Big].$$

$p=96$, $h=91.7=92$ say, $B=94$, $B'=83$.

$\therefore t=\pi/C\,(4.3093+90.3161+409.9964)=1.76848\pi=5$ hours, 33 minutes, 21 seconds.

The actual time observed for filling this tank to within 4 inches of the top was $5\frac{1}{2}$ hours.

MISCELLANEOUS.

149. Proposed by F. P. MATZ, Ph. D., Sc. D.

Given $\sin^{-1}u+\sin^{-1}\frac{1}{2}u=\frac{1}{4}\pi$, to find u.

Solution by J. EDWARD SANDERS.

By use of the addition theorem, we have

$$\tfrac{1}{2}\sqrt{2}=u.\tfrac{1}{2}\sqrt{(4-u^2)}+\tfrac{1}{2}u.\sqrt{(1-u^2)}.$$

Squaring twice and arranging, we get the trinomial $17u^4-20u^2=-4$, or $u=\pm\sqrt{(\frac{10}{17}\pm\frac{4}{17}\sqrt{2})}$. Whence $u=\pm.50544945$......... or $\pm.95968298$..........

The first value is the one solving the question.

Also solved by R. D. Carmichael, G. W. Greenwood, A. H. Holmes, L. E. Newcomb, J. Scheffer, W. L. Tryon, G. B. M. Zerr, and the Proposer.

150. Proposed by T. N. HAUN, Mohawk, Tenn.

If $\dfrac{\sin\phi}{\sin\psi}=m$, find maximum and minimum value of $\dfrac{\sin(\phi+\theta)}{\sin(\psi+\theta)}$, where θ is known.

I. Solution by A. H. HOLMES.

$\dfrac{\sin\phi}{\sin\psi}=m.\quad \therefore \sin\phi=m\sin\psi$ and $\cos\phi=\sqrt{(1-m^2\sin^2\psi)}$.

$$\therefore\ \frac{m\sin\psi\cos\theta+\sin\theta\sqrt{(1-m^2\sin^2\psi)}}{\sin\psi\cos\theta+\sin\theta\sqrt{(1-\sin^2\psi)}}=\text{maximum or minimum.}$$

Differentiating, etc.,

$$\sin^4\psi-\frac{2\cos^2\theta(m^2+\sin^2\theta+\cos^2\theta)}{(m^2-\sin^2\theta+\cos^2\theta)^2+4\sin^2\theta\cos^2\theta}$$

$$=-\frac{\cos^4\theta}{(m^2-\sin^2\theta+\cos^2\theta)^2+4\sin^2\theta\cos^2\theta}.$$

$$\therefore \sin\psi=\frac{\cos\theta}{\sqrt{(m^2-2m\sin\theta+1)}}\text{ for maximum,}$$

$$\text{and }\sin\psi=\frac{\cos\theta}{\sqrt{(m^2+2m\sin\theta+1)}}\text{ for minimum.}$$

$\therefore$ Maximum value of $\dfrac{\sin(\phi+\theta)}{\sin(\psi+\theta)}=\dfrac{m+\sin\theta}{1+m\sin\theta}$ when $\theta+\phi+\psi=\frac{1}{2}\pi$; and similarly=minimum when $\theta+\phi+\psi=3\pi/2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D.

$\dfrac{\sin\phi}{\sin\psi}=m$, $\dfrac{\sin(\phi+\theta)}{\sin(\psi+\theta)}=u$=maximum and minimum.

$\therefore \cos\phi\ \sin\psi\ d\phi-\cos\psi\ \sin\phi\ d\psi=0$.

$\cos(\phi+\theta)\sin(\psi+\theta)d\phi-\cos(\psi+\theta)\sin(\phi+\theta)d\psi=0$.

$\therefore \tan(\psi+\theta)\tan\phi=\tan\psi\ \tan(\phi+\theta)$.

$(\tan\phi-\tan\psi)(1-\tan\phi\ \tan\psi-\tan\theta\ \tan\phi-\tan\theta\ \tan\psi)=0$.

$\therefore \phi=\psi$ and $\theta+\phi+\psi=\pi/2$ or $3\pi/2$. ϕ cannot$=\psi$ unless m=unity.

$\therefore \dfrac{\sin(\phi+\theta)}{\sin(\psi+\theta)}=\dfrac{\cos\psi}{\cos\phi}=u$.

$\sin\phi=m\sin\psi=m\cos(\phi+\theta)$ or $-m\cos(\phi+\theta)$.

$\therefore \tan\phi=\dfrac{m\cos\theta}{1+m\sin\theta}$ or $\dfrac{m\cos\theta}{m\sin\theta-1}$.

$m\sin\psi=\cos(\psi+\theta)$ or $-\cos(\psi+\theta)$.

$\therefore \tan\psi=\dfrac{\cos\theta}{m+\sin\theta}$ or $\dfrac{\cos\theta}{\sin\theta-m}$.

$\therefore u=\dfrac{m+\sin\theta}{1+m\sin\theta}$=maximum when $\theta+\phi+\psi=\pi/2$.

$u=\dfrac{\sin\theta-m}{m\sin\theta-1}$=minimum when $\theta+\phi+\psi=3\pi/2$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

244. Proposed by ELMER SCHUYLER.

Solve $(x+1)(y-1)=a(x-1)(y+1)$; $(x^5-1)(y-1)=b^2(y^5-1)(x-1)$.

245. Proposed by SAM I. JONES, A. B., Gunter Bible College, Gunter, Texas.

The shell of a hollow iron ball is 4 inches thick, and contains $\frac{1}{5}$ of the number of cubic inches in the whole ball. Find the diameter of the ball.

246. Proposed by J. EDWARD SANDERS.

Find in terms of the roots the area common to the two curves $x+y^2=a$ $(=\frac{70}{9})$ and $x^2+y=b(=\frac{28}{9})$. What are the conditions that the roots are all real? Are there values of a and b that will make all the roots rational?

AVERAGE AND PROBABILITY.

172. Proposed by J. EDWARD SANDERS.

A circular arc, with center at one corner of a given square, is drawn through a point at random in the square. What is the average length of the arc within the square?

172. Proposed by J. EDWARD SANDERS.

What is the average length of all straight lines that can be drawn within a given triangle?

GEOMETRY.

268. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find, without the aid of trigonometry, the side of an inscribed regular polygon of $2n$ sides, if the side of an inscribed regular polygon of n sides is 16 feet. [Wentworth's *Plane Geometry*, Revised Edition, problem 512, page 244.]

269. Proposed by J. SCHEFFER, A. M.

Find the area of a segment, if the chord of the segment is 10 feet, and the radius of the circle is 16 feet.

270. Proposed by F. R. HONEY, Ph. B., Hartford, Conn.

What portion of the heavens is always invisible to an observer whose latitude is given?

271. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two equal concentric ellipses have their axes at an angle θ. Find the area of the quadrilateral circumscribing both, in terms of θ and the semi-axes.

272. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A point A revolves with uniform speed in a circle. A point B revolves around A, at a uniform distance from it, with the same angular velocity, but in the opposite direction. Determine the locus of B.

273. Proposed by A. H. HOLMES, Brunswick, Maine.

Required a purely geometrical solution of the problem, to find the contents of a solid generated by the revolution of a semi-segment of a circle about the sine of its arc.

GROUP THEORY.

11. Proposed by SAUL EPSTEEN, Chicago, Ill.

Find the six-parameter continuous group which leaves invariant the surface of second order $x_1x_2-x_3x_4=0$.

CALCULUS.

207. Proposed by F. P. MATZ, Sc. D., Ph. D.

If K represents the complete elliptic integral of the first kind, prove that

$$\int_0^1 \frac{K d\kappa}{1+\kappa} = \tfrac{1}{4}\pi^2.$$

208. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0.$$

MECHANICS.

185. Proposed by J. EDWARD SANDERS.

A perfectly flexible rope whose weight is w per linear unit, and length $2l$, rests in equilibrium on a smooth peg. If now one end be raised a distance a and then released, find the time in which this end will rise to the height x above its original position, and the tension at that instant of the rope at the point where it passes over the peg.

MISCELLANEOUS.

152. Proposed by J. EDWARD SANDERS.

A conductor, the equation of the surface of which is

$$\frac{x^2}{25}+\frac{y^2}{16}+\frac{z^2}{9}=1,$$

is charged with 80 units of electricity, what is the density at a point for which $x=3$, $y=3$? If the density of this point be a, what is the whole charge on the ellipsoid? [From Peirce's *Potential Functions*, example 165, p. 388.]

153. Proposed by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

Two men start from Columbus, Ohio, at the same time; one travels east and the other west. They travel at the rate of 4 miles an hour from sunrise to sunset each day until they meet. Where will they meet and what distance will each have traveled?

154. Proposed by D. BIDDLE (Unsolved problem in the Educational Times, London).

Prove that the proper angle at which to cross a street when a person wishes to continue his course on the other side, and the roadway is n times as muddy as the pavement, is that of which the sine is $(n^2-1)/(n^2+1)$.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns):

Average and Probability, 167. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

Geometry, 246. Proposed by T. L. CROYES, Paris, France.

Given a movable point O on a fixed diameter of a circle S, an inscribed triangle ABC, and the perpendiculars OM, ON, OP from the point O on the sides AB, AC, BC. Prove, by pure geometry, that the circle circumscribing the triangle MNP will always pass through a fixed point.

Group Theory, 8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the eight players being paired in each round, every pair to be matched once and but once in the tournament. List the possible programs different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

Mechanics, 181. Proposed by F. ANDEREGG, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

A triangle AOB, of which the sides, OA, AB, and the angle at O are a, b, and α, revolves uniformly about O, so that OA makes the angle nt with the axis of x, and carries a circle of which AB is the diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe an ellipse whose axes are

$$\sqrt{(a^2+b^2+2ab\cos\alpha)}\pm\sqrt{(a^2+b^2-2ab\cos\alpha)}.$$

NOTES.

Dr. O. L. Underhill has been appointed instructor in mathematics at Princeton University.

Mr. A. D. Pitcher has been appointed fellow in mathematics at the University of Kansas.

Mr. D. A. Lehman has been appointed instructor in mathematics at the University of Wisconsin.

Mr. R. F. Sharpe, assistant in mathematics at Cornell University, is Second Wrangler in mathematics at Cambridge, England.

Dr. Oswald Veblen and Dr. J. W. Young have been appointed to assistant professorships in mathematics at Princeton University.

Dr. O. D. Kellogg, of Princeton University, has been appointed assistant professor of mathematics at the University of Missouri.

Mr. R. L. Borger, of the University of Florida, has been appointed instructor in mathematics at the University of Missouri.

The assistants in mathematics at Cornell University for 1905-1906 are F. C. Edminster, R. F. Sharpe, E. C. Colpitts, W. M. Carruth.

The following have been appointed fellows in mathematics at The University of Chicago: George D. Birkhoff, Louis Angold, William R. Longley.

The student assistants in mathematics at the University of Kansas for the year 1905-1906 are Mr. U. G. Mitchell, Miss Birdie Greenough, Miss Frances Lahmir and Miss Mabel Davis.

At the University of Kansas, Dr. H. B. Newson has been promoted from an associate professorship to a full professorship in mathematics, and Mr. J. A. G. Shirk has been appointed instructor in mathematics.

Mr. P. P. Boyd has been appointed Erastus Brooks Fellow in Mathematics, and Mr. C. F. Craig University Scholar in mathematics at Cornell University. The Oliver Scholarship was not awarded this year.

Miss Sarah Elizabeth Cronin, M. S., lately fellow in mathematics in the State University of Iowa, has been appointed instructor in mathematics in the Iowa State College of Agriculture and Mechanic Arts at Ames.

At the Northwestern University, Professor Thomas F. Holgate has been appointed to the Noyes Professorship of Mathematics, Dr. D. R. Curtiss has been appointed assistant professor of mathematics, and Dr. J. C. Morehead has been appointed instructor in mathematics.

Dr. J. V. Westfall and Mr. W. E. Beck have resigned their positions in the department of mathematics in the State University of Iowa; the former to accept an actuarial position with the firm of Haskins & Sells of New York City, the latter in response to a call to service in the United States Coast and Geodetic Survey. The positions thus made vacant have been filled by the appointment of Mr. R. P. Baker and Mr. C. M. Thorne as instructors.

We learn from *Science* that twenty doctorates in mathematics were conferred by American universities in the year 1905. This is an increase of seven over the year 1904. Following are the names, the institution conferring the degree, and title of the thesis in each case:

Clark University. Reginald Bryant Allen, "On Hypercomplex Number Systems Belonging to an Arbitrary Domain of Rationality;" Charles E. Browne, "A Study of the Simpler Arithmetic Processes;" John Shaw French, "On the Theory of the Pertingents to a Plane Curve;" Jesse N. Gates, "Cubic and Quar-

tic Surfaces in Fourfold Space;" Herbert G. Keppel, "The Cubic Three-Spread Ruled with Planes in Four-fold Space."

The University of Chicago. Herbert Edwin Jordan, "Group Characters of Various Types of Linear Groups;" Thomas E. McKinney, "Concerning a Certain Type of Continued Fractions Depending upon a Variable Parameter;" Robert L. Moore, "Sets of Metrical Hypotheses of Geometry;" Arthur Whipple Smith, "The Symbolic Treatment of Differential Geometry."

The University of Pennsylvania. Oliver Edmunds Glenn, "The Determination of the Abstract Groups of Order p^2qr; p, q, and r being Distinct Primes;" Ulysses Sherman Hanna, "The Bitangentials of the Plane Quintic and Plane Sextic;" Alice Madeleine McKelden, "Groups of Order 2^m that Contain Cyclic Subgroups of Order 2^{m-1}, 2^{m-2}, and 2^{m-3}."

Johns Hopkins University. Henry Bayard Phillips, "Some Invariants and Covariants of Ternary Collineations;" Roswell P. Stephens, I. "On a Curve of the Fifth Class," II. "On a System of Parastroids."

Yale University. Raymond B. McClenon, "On Simple Integrals with Variable Limits;" James Caddall Morehead, "Numbers of the Form $2q=1$ and Fermats Numbers."

Harvard University. Walter Burton Ford, "On a Problem of Analytic Extension as Applied to Functions Defined by Power Series."

Princeton University. Adam M. Hiltebeitel, "The Problem of Two Fixed Centers and Certain of its Generalizations."

Leland Stanford University. William A. Manning, "Studies on the Class of Primitive Substitution Groups."

Cornell University. Oscar P. Akers, "On the Congruence of Axes in a Bundle of Linear Complexes."

BOOKS AND PERIODICALS.

The Continuum as a Type of Order: An Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers. By E. V. Huntington. Reprinted from the *Annals of Mathematics*, (2), Vol. 6, pp. 151-184, and Vol. 7, pp. 15-43. Price, $0.50. The Publication Office of Harvard University.

This reprint gives in 63 pages a systematic elementary exposition of the Dedekind-Cantor theory. The definitions and concepts are presented with great care; numerous simple examples are given to illustrate the systems which have, and those which have not, the property considered. The author is to be congratulated on his success in presenting this important subject in such an attractive and readable form. D.

An Elementary Treatise on Graphs. By George A. Gibson, M. A., F. R. S. E. The MacMillan Co. Price, 3*s*, 6*d*.

Mr. Gibson's book deals with this branch of mathematical science in a manner at once thorough, connected, and attractive. The earlier portions are presented in a form

suitable even for the beginner, whilst the more advanced student will find the treatise adequate for much of his higher work. It discusses the subject in many of its principal developments and considers those amongst its numerous applications which are of primary interest and admit of discussion by means of elementary mathematics. Many and varied problems in geometrical and physical science and others based on statistics are either treated fully for the sake of illustration or proposed as exercises for the student. In the course of his brief preface the author dwells on the importance of assigning its true position to graphical work. Until very lately but little general use was made of graphical methods and there is little likelihood that in this country, at the present time, the graph will receive too much attention by teachers, at the expense of analysis. Mr. Gibson's caution against this is, however, appropriate. As usual, what is desirable is the happy medium—thc recognition that, in a sense, graphical representation and analysis may, perhaps, be regarded as complementary instruments of investigation. Each is powerful for special purposes, and a problem is often best understood if considered from the two standpoints. Mr. Gibson's work will prove a thoroughly acceptable text-book, supplying a definite want. G.

Forces Due to Eccentric Weights Attached to Rolling Wheels. By Professor Calvin M. Woodward, Ph. D. Reprinted from *Journal of Association of Engineering Societies*, Vol. XXXV, No. 1.

A noteworthy piece of research on the motion of bodies constrained to move in trochoids. G.

The American Journal of Mathematics. Published under the auspices of Johns Hopkins University. Edited by Frank Morley and other mathematicians.

The October number contains the following contributions: Concerning Certain 4-Space Quintic Configurations of Point Ranges and Congruences, and their Analogues in Ordinary Space, by C. J. Keyser. Some Relations between Group Theory and Number Theory, by G. A. Miller. The Differential Invariants of Space, by J. E. Wright. An Arithmetic Treatment of Some Problems in Analysis Situs, by L. D. Ames. On the Definition of Reducible Hypercomplex Number Systems, by H. B. Leonard.

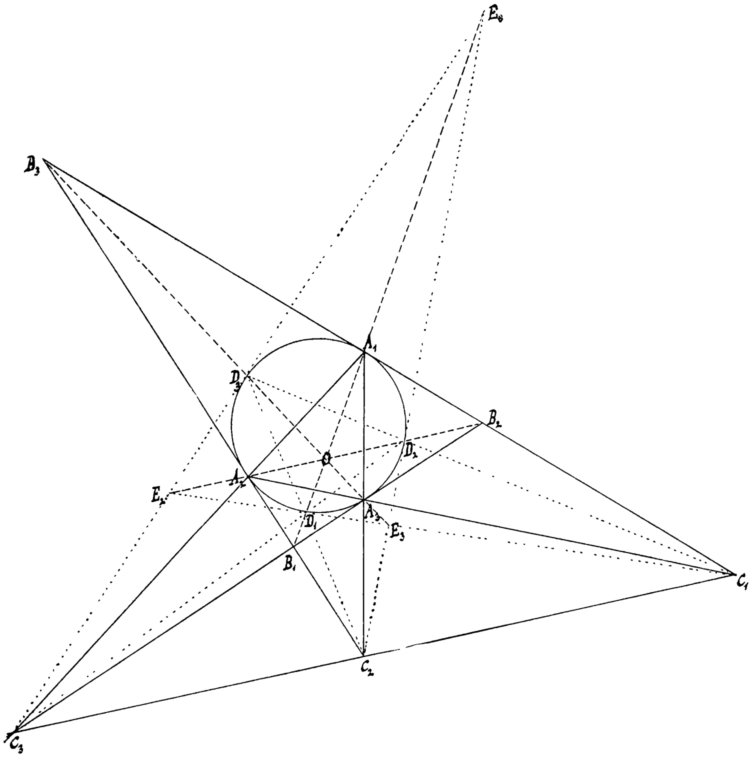
The Annals of Mathematics. Published under the auspices of Harvard Uuiversity.

The October number contains the following papers: Concerning Green's Theorem and The Cauchy-Riemann Differential Equations, by M. B. Porter. On the Singularities of Tortuous Curves, by Paul Saurel. On the Twist of a Tortuous Curve, by Paul Saurel. The Continuum as a Type of Order, Chapters V and VI, by E. V. Huntington. A Problem in Analytic Geometry, with a Moral, by Maxime Böcher.

The following periodicals have been received: The Scientific American, The Educational Times, The Nation, The Review of Reviews, The Literary Digest, Ohio Educational Monthly, Sierra Educational News, The Ohio Teacher, The Physical Review, Bulletin of American Mathematical Society, School Science and Mathematics, Proceedings of the London Mathematical Society, The Open Court, The School Visitor, Popular Astronony, L'Enseignement Mathematique, Bollettino della Associazione.

Errata.

Page 64, line 10 from bottom, for "superficial" read "artificial."
Page 161, line 4, should read, $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{144}+\frac{1}{840}+\dots\dots$



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A USEFUL DIAGRAM FOR EXAMPLES IN MODERN ANALYTIC GEOMETRY.

By PROFESSOR H. MASCHKE.

In the following brief article I discuss, from different points of view, a certain diagram, which leads to a great number of examples and exercises in modern analytic geometry.

Let A_1 be a point on the conic

$$x_1x_2+r_2x_3+x_3x_1=0.$$

Designate its trilinear coördinates x_1, x_2, x_3 by α, β, γ, respectively, and write briefly

$$A_1=(\alpha,\ \beta,\ \gamma).$$

Then the points

$$A_2=(\beta,\ \gamma,\ \alpha) \text{ and } A_3=(\gamma,\ \alpha,\ \beta)$$

will lie on the same conic. They form with A_1 a triangle which shall be denoted by $[\alpha, \beta, \gamma]$.

According to Pascal's theorem the intersections C_1, C_2, C_3 of the tangents to the conic at A_1, A_2, A_3 and the respective opposite sides of the triangle lie on a straight line ω.

The equation of the tangent at A_1 is

$$(\beta+\gamma)x_1+(\gamma+\alpha)x_2+(\alpha+\beta)x_3=0,$$

and the coördinates of $C_1=(\beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta)$. The coördinates of C_2 and C_3 are obtained from those of C_1 by a cyclic permutation of α, β, γ. Hence the line ω has the equation

$$x_1+x_2+x_3=0,$$

and its pole O the coördinates (1, 1, 1).

The line ω is, therefore, independent of the selection of the point A_1 on the conic. All the triangles $[\alpha, \beta, \gamma]$ have one and the same Pascal-line ω, and every triangle $[\alpha, \beta, \gamma]$ determines on ω a triple of points C_1, C_2, C_3.

If we select both the conic section and the line ω arbitrarily (not, however, intersecting in real points), then every triangle satisfying the conditions of the diagram is a triangle $[\alpha, \beta, \gamma]$. In other words: If we take as triangle of reference one of the inscribed triangles and the line ω as unit-line (or its pole O as unit-point) and if further we choose the point $A_1=(\alpha, \beta, \gamma)$ arbitrarily on the conic, then the coördinates of A_2 and A_3 are found to be (β, γ, α) and (γ, α, β).

Every point-triple C_1, C_2, C_3 is determined by one of its points, C_1 for instance, just as every triangle $[\alpha, \beta, \gamma]$ is determined by one of its vertices, *e. g.* A_1. Every point C_1, however, while it determines C_2 and C_3 uniquely, determines two triangles $[\alpha, \beta, \gamma]$, namely, A_1, A_2, A_3 and D_1, D_2, D_3, corresponding to the two tangents which can be drawn from C_1 to the conic.

The coördinates of all the lines and points shown in the diagram are all given by simple expressions in terms of α, β, γ and can easily be deduced by always using the relation

$$\alpha\beta+\beta\gamma+\gamma\alpha=0$$

for simplification.

As examples I give the coördinates of the points

$$B_1=(\alpha-\beta-\gamma,\ \beta-\gamma-\alpha,\ \gamma-\alpha-\beta),$$
$$D_1=(2\beta+2\gamma-\alpha,\ 2\gamma+2\alpha-\beta,\ 2\alpha+2\beta-\gamma),$$
$$E_1=(\beta+\gamma-5\alpha,\ \gamma+\alpha-5\beta,\ \alpha+\beta-5\gamma).$$

The line ω meets the conic in the two imaginary points $1, \varepsilon, \varepsilon^2$ and $1, \varepsilon^2, \varepsilon$, where

$$\varepsilon=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}.$$

These two imaginary points ω_1 and ω_2, together with any point-triple C_1, C_2, C_3 form a so-called cyclic-projective system of points. That means the following three ranges of five points are projective:

$$(\omega_1, C_1, C_2, C_3, \omega_2),\quad (\omega_1, C_2, C_3, C_1, \omega_2),\quad (\omega_1, C_3, C_1, C_2, \omega_2).$$

Since the two triangles B_1, B_2, B_3 and E_1, E_2, E_3 are circumscribed about one and the same conic, the six vertices B and E must lie on a conic. The equation of this conic is

$$x_1^2+x_2^2+x_3^2+3(x_1x_2+x_2x_3+x_3x_1)=0.$$

To this conic the triangle B_1, B_2, B_3 (or E_1, E_2, E_3) bears the same relation as the original triangle A to the original conic, *i. e.* the tangents to the conic at B_1, B_2, B_3 (or E_1, E_2, E_3) pass through C_1, C_2, C_3, respectively.

Likewise the conic

$$x_1^2+x_2^2+x_3^2-2(x_1x_2+x_2x_3+x_3x_1)=0$$

is inscribed in the two triangles A and D. The lines joining the contact points of these two triangles pass again through C_1, C_2, C_3. In this way we obtain an infinite chain of conics Γ. All these conics Γ pass through the same points ω_1, ω_2 on the line ω, and since the point O is the pole of ω with respect to every conic, the conics Γ are tangent to each other at ω_1 and ω_2.

If we transform by projection the original conic into a circle and simultaneously the line ω into the line at infinity, which is always possible, then the conics Γ are projected into conics passing through the two circular points at infinity where they are tangent to each other, *i. e.* into concentric circles. The inscribed and circumscribed triangles of the diagram are transformed into equilateral triangles.

From this system of concentric circles with inscribed and circumscribed triangle all the properties of the diagram involving the conics Γ can now easily be deduced.

If we take as original conic an ellipse and as line ω the line at infinity, the point O becomes the center of the ellipse and also the point of intersection of the medians of every triangle A_1, A_2, A_3, and it can easily be shown not only that all these triangles are of equal area, but also that this area represents the maximum value of the areas of all triangles inscribed into the ellipse.

THE UNIVERSITY OF CHICAGO, *November, 1905.*

A PERFECT MAGIC SQUARE.

By PROFESSOR F. ANDEREGG.

In the accompanying magic square, the sum of the numbers in any line, column, or diagonal is the same (2056). Instead of continuous lines, any four numbers can be taken from the first and third groups of four numbers, and any four from the second and fourth groups in the first line, and the corresponding numbers in the last line. Similar combinations can be made for the second and fifteenth lines, the third and fourteenth, etc. Similar combinations can also be made for the columns.

Instead of a complete diagonal two incomplete diagonals can be taken on

opposite sides of either complete diagonal so that the two together contain sixteen numbers.

The sum of the sixteen numbers in any four two-squares symmetrically situated with respect to the center of the sixteen-square is 2056. The sum of the sixty-four numbers in any eight-square taken at random, or of any four four-squares symmetrically situated with respect to the center of the sixteen-square is 8224.

1	2	3	4	248	247	246	245	73	74	75	76	192	191	190	189
17	18	19	20	232	231	230	229	89	90	91	92	176	175	174	173
33	34	35	36	216	215	214	213	105	106	107	108	160	159	158	157
49	50	51	52	200	199	198	197	121	122	123	124	144	143	142	141
128	127	126	125	137	138	139	140	56	55	54	53	193	194	195	196
112	111	110	109	153	154	155	156	40	39	38	37	209	210	211	212
96	95	94	93	169	170	171	172	24	23	22	21	225	226	227	228
80	79	78	77	185	186	187	188	8	7	6	5	241	242	243	244
181	182	183	184	68	67	66	65	253	254	255	256	12	11	10	9
165	166	167	168	84	83	82	81	237	238	239	240	28	27	26	25
149	150	151	152	100	99	98	97	221	222	223	224	44	43	42	41
133	134	135	136	116	115	114	113	205	206	207	208	60	59	58	57
204	203	202	201	61	62	63	64	132	131	130	129	117	118	119	120
220	219	218	217	45	46	47	48	148	147	146	145	101	102	103	104
236	235	234	233	29	30	31	32	164	163	162	161	85	86	87	88
252	251	250	249	13	14	15	16	180	179	178	177	69	70	71	72

OBERLIN COLLEGE, *April 26, 1905.*

DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

SOLUTIONS OF PROBLEMS.

The following problem was received too late for publication: Miscellaneous No. 150, solved by L. E. Newcomb.

ALGEBRA.

243. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Find the infinite root of $\frac{1}{x}+\frac{1}{a}=\sqrt{\left[\frac{1}{a^2}-\sqrt{\frac{1}{a^2x^2}+\frac{1}{x^4}}\right]}$.

Solution by ELMER SCHUYLER, A. B., Brooklyn, New York.

Squaring the sides and reducing we have,

$$\frac{1}{x^2}+\frac{2}{ax}=-\sqrt{\frac{1}{a^2x^2}+\frac{1}{x^4}}.$$

Whence $1/x=0$, or $x=\infty$. Also,

$$\frac{1}{x}+\frac{2}{a}=-\sqrt{\frac{1}{a^2}+\frac{1}{x^2}}.$$

Squaring this and reducing, we get

$$\frac{1}{x}=-\frac{3}{4a},\text{ or } x=-\frac{4a}{3}.$$

Since $\frac{1}{x^2}$ cancelled out on each side of the equation, $\frac{1}{x}=0$, or $x=\infty$. The only value of x satisfying the original equation, the radicals being taken with positive signs, is $x=\infty$.

Also solved by B. F. Finkel, G. W. Greenwood, J. E. Sanders, G. B. M. Zerr, and the Proposer.

AVERAGE AND PROBABILITY.

168. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average area of a triangle two of whose sides have the constant sum $2a$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x, y be the sides and θ the included angle.

Then $\frac{1}{2}xy\ \sin\theta$=area, and $x+y=2a$.

$$\therefore \text{Average area} = \triangle = \frac{\frac{1}{2}\int_0^{2a}\int_0^{\pi} x(2a-x)\sin\theta\ dx\, d\theta}{\int_0^{2a}\int_0^{\pi} dx\, d\theta}.$$

$$\therefore \triangle = \frac{1}{4\pi a}\int_0^{2a}\int_0^{\pi} x(2a-x)\sin\theta\, dx\, d\theta = \frac{1}{2\pi a}\int_0^{2a} x(2a-x)dx = \frac{2a^2}{3\pi}.$$

169. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square?

I. Solution by B. F. FINKEL, A. M., Fellow in The University of Pennsylvania.

Let $ABCD$ be the given square, the length of whose side is a; and let EF and $E'F'$ be two parallel lines: EF terminating in two adjacent sides of the square, and $E'F'$ in two opposite sides. Let AH, $=x$, be the perpendicular to EF from A, and let the angle $FAH=\theta$.

Then $EF=\dfrac{x}{\sin\theta\ \cos\theta}$, and $E'F'=\dfrac{a}{\cos\theta}$.

Since the law of distribution of the lines is not given, we will assume that those having a given direction are uniformly distributed, and those passing through a given point are distributed so that the number of lines lying in a given angle is directly proportional to the magnitude of the angle.

Now, the limits of x are 0 and $AH'=a\sin\theta$ for $EF=x/\sin\theta\ \cos\theta$, and $AH'=a\sin\theta$ and $AH''=a\cos\theta$, for $EF=E'F'=a/\cos\theta$; and the limits of θ are 0 and $\frac{1}{4}\pi$. Then the average length of lines drawn within the square is

$$l_\triangle = \frac{\int_0^{\frac{1}{4}\pi} d\theta \left[2\int_0^{a\sin\theta} \sec\theta\ \cos\theta\ x dx + \int_{a\sin\theta}^{a\cos\theta} a\sec\theta\ dx\right]}{\int_0^{\frac{1}{4}\pi} d\theta \int_0^{\sqrt{2}\,a\cos(\frac{1}{4}\pi-\theta)} dx}$$

$$= \frac{1}{a}\int_0^{\frac{1}{4}\pi} d\theta \left[2\int_0^{a\sin\theta} \sec\theta\csc\theta\, x dx + a\int_{a\sin\theta}^{a\cos\theta} \sec\theta\ dx\right]$$

$$= \frac{1}{a}\int_0^{\frac{1}{4}\pi} d\theta \left[\sec\theta\csc\theta\Big(x^2\Big)_0^{a\sin\theta} + a\sec\theta\Big(x\Big)_{a\sin\theta}^{a\cos\theta}\right]$$

$$= \frac{1}{a}\int_0^{\frac{1}{4}\pi} a^2\, d\theta = \tfrac{1}{4}\pi a.$$

II. Solution* by G. B. M. ZERR. A. M., Ph. D., Parsons, W. Va.

Let $ABCD$ be the given square. Let F be a point in AB, E a point in AD, and G, H, points in DC. Let PQ be a length, measured at random in FE and FG. Let $AF=x$, $AE=y$, $PQ=z$, $DH=u$, $DG=v$, $AB=a$, $\sqrt{(x^2+y^2)}=z'$, $\sqrt{[a^2+(u-v)^2]}=z''$. M=the average length.

$$\therefore M=\frac{\int_0^a\int_0^a\int_0^{z'} z\,dx\,dy\,dz+\int_0^a\int_0^u\int_0^{z''} z\,du\,dv\,dz}{\int_0^a\int_0^a\int_0^{z'} dx\,dy\,dz+\int_0^a\int_0^u\int_0^{z''} du\,dv\,dz}$$

$$=\frac{\int_0^a\int_0^a(x^2+y^2)dx\,dy+\int_0^a\int_0^u[a^2+(u-v)^2]du\,dv}{2\int_0^a\int_0^a\sqrt{(x^2+y^2)}dx\,dy+2\int_0^a\int_0^u\sqrt{[a^2+(u-v)^2}du\,dv}$$

$$=\frac{\int_0^a a(3x^2+a^2)dx+\int_0^a(3a^2u+u^3)du}{3\int_0^a\left[a\sqrt{(a^2+x^2)}+x^2\log\left(\frac{a+\sqrt{(a^2+x^2)}}{x}\right)\right]dx+3\int_0^a\left[u\sqrt{(a^2+u^2)}+a^2\log\left(\frac{u+\sqrt{(a^2+u^2)}}{a}\right)\right]du}$$

$$=\frac{11a}{4[2+\sqrt{2}+5\log(1+\sqrt{2})]}.$$

Also solved by J. E. Sanders.

171. Proposed by O. E. GLENN, A. M., Ph. D., Drury College.

There are n derelict steamers afloat in a circular sea of radius r. The water in the sea is moving northward in a current whose velocity varies inversely as the perpendicular distance from the north-south tangent to the sea on its west beach. Find the probability that a ship crossing the sea on a random diameter will encounter e derelicts during the voyage.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the sea, POQ the random diameter, G the position of a derelict when the start is made at P, AOB the diameter perpendicular to the north-south tangent AC on the west beach A. Let FE be the chord through G perpendicular to AB, cutting AB in L and PQ in K, H the foot of the perpendicular from Q on AB. Let $\angle QOB=\theta$, $AL=z$, $GK=y$, u=uniform velocity of ship on PQ, p=chance of meeting one derelict. Then $OK=OL\sec\theta=(z-r)\sec\theta$, velocity of derelict$=m/z$.

*The two methods illustrated are typical of the different results to be obtained when different laws of distribution are assumed.

$$\therefore \frac{r+(z-r)\sec\theta}{u}=\frac{yz}{m}. \qquad \therefore y=\frac{m}{uz}(r-r\sec\theta+z\sec\theta)=y'.$$

The limits of θ are 0 and 2π; of z, 0 and $2r$ for total, and $r(1+\cos\theta)=z'$ to $r(1-\cos\theta)=z''$ for favorable cases; of y, 0 and $2\sqrt{(2rz-z^2)}=y''$ for total, and 0 and y' for favorable cases.

$$\therefore p=\frac{\int_0^{2\pi}\int_{z''}^{z'}\int_0^{y'}d\theta\, dz\, dy}{\int_0^{2\pi}\int_0^{2r}\int_0^{y''}d\theta\, dz\, dy}=\frac{1}{2\pi^2r^2}\int_0^{2\pi}\int_{z''}^{z'}\int_0^{y'}d\theta\, dz\, dy$$

$$=\frac{m}{2\pi^2r^2u}\int_0^{2\pi}\int_{z''}^{z'}\frac{(r-r\sec\theta+z\sec\theta)d\theta\, dz}{z}$$

$$=\frac{m}{\pi^2ru}\int_0^{2\pi}(1+\sec\theta\log\tan\tfrac{1}{2}\theta-\log\tan\tfrac{1}{2}\theta)d\theta=\frac{2m}{\pi ru}.$$

Therefore the chance of meeting one derelict out of the n is $\Sigma p=2nm/\pi ru$, and the probability that e will be encountered is

$$\Pi\left(\frac{2nm}{\pi ru}\right)=\left(\frac{2nm}{\pi ru}\right)^e.$$

CALCULUS.

203. Proposed by S. A. COREY, Hiteman, Iowa.

Evaluate* $\int_0^{\pi}\frac{\sin mx\, dx}{x}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^{\pi}\frac{\sin mx}{x}dx=\int_0^{\pi}\left(m-\frac{m^3x^2}{3!}+\frac{m^5x^4}{5!}-\frac{m^7x^6}{7!}+\text{.........}\right)dx$$

$$=m\pi\left(1-\frac{m^2\pi^2}{3.3!}+\frac{m^4\pi^4}{5.5!}-\frac{m^6\pi^6}{7.7!}+\text{.........}\right).$$

Also solved by L. E. Newcomb.

204. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Required the variation of $\int Vdx$ where V is a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ ..... and v where $v=\int V'dx$ and V' is also a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.....

*See Byerly's *Integral Calculus* (p. 23, Table of Integrals) Formulae 211 and 241. A solution not in the form of an infinite series would also be desirable.

*Solution by the PROPOSER.

The variation of $\int Vdx$ may be divided into $\triangle u$ and $\triangle u_1$, the former arising supposing v constant, the latter from the variation of v. Thus $\triangle u = H + \int K\triangle ydx$, $\triangle u_1 = \int \frac{dV}{dv}\triangle vdx$. If the letters represent for V what their primes represent for V', $\triangle u = \int(N'\triangle y + P'\triangle p + \text{.........})dx$. If $L = \frac{dV}{dv}$ and $I = \int Ldx$,

$$\triangle u_1 = \int L\triangle vdx = Idv - \int I\frac{d\triangle v}{dx}dx = I(H' + \int K'\triangle ydx) - (Hi' + \int Ki'\triangle ydx)$$

(Hi, Ki denoting H' and K' when IN', IP', etc., are substituted for N', P',.....
Then

$$\triangle u + \triangle u_1 = \triangle y\left(P - \frac{dQ}{dx}\cdots\right) + \triangle p\left(Q - \frac{dR}{dx} + \cdots\right) + \cdots\cdots$$

$$+\int\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \cdots\right)\triangle ydx + I\triangle y\left(P' - \frac{dQ'}{dx} + \frac{d^2R'}{dx^2}\cdots\right)$$

$$+I\triangle p\left(Q' - \frac{dR'}{dx} + \cdots\right) + \cdots + I\int\left(N' - \frac{dP'}{dx} + \frac{d^2Q'}{dx^2}\cdots\right)dx$$

$$-\triangle y\left(IP' - \frac{dIQ'}{dx} + \cdots\right) - \triangle p\left(IP' - \frac{dIQ'}{dx} + \cdots\right) - \cdots\cdots$$

$$-\int\left(IN' - \frac{dIP'}{dx} + \frac{d^2IQ'}{dx^2}\cdots\right)dx.$$

Also solved by G. B. M. Zerr.

205. Proposed by Z. T. JACKSON, St. Louis, Mo.

Evaluate† $\int_0^{\frac{1}{2}\pi} \log \sin x \, dx$.

Solution by J. E. SANDERS, Reinersville, Ohio.

$$u = \int_0^{\frac{1}{2}\pi} \log \sin x \, dx = \int_0^{\frac{1}{2}\pi} \log \sin(\tfrac{1}{2}\pi - x)dx = \int_0^{\frac{1}{2}\pi} \log \cos x \, dx.$$

$$\therefore 2u = \int_0^{\frac{1}{2}\pi} (\log \sin x + \log \cos x)dx = \int_0^{\frac{1}{2}\pi} \log(\sin x \cos x)dx$$

$$= \int_0^{\frac{1}{2}\pi} \log \frac{\sin 2x}{2} dx = \int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx - \frac{\pi}{2}\log 2.$$

*See Williamson's *Integral Calculus*, Sixth Edition, p. 275.
†Byerly, *Integral Calculus*, p. 102.

Let $2x=x'$, then

$$\int_0^{\frac{1}{2}\pi}\log\sin 2x\,dx=\tfrac{1}{2}\int_0^{\pi}\log\sin x'\,dx'=\int_0^{\frac{1}{2}\pi}\log\sin x\,dx.$$

$$\therefore 2u=u-\frac{\pi}{2}\log 2,\quad u=-\frac{\pi}{2}\log 2=\frac{\pi}{2}\log\tfrac{1}{2}.$$

Also solved by M. E. Graber, G. W. Greenwood, L. E. Newcomb, and G. B. M. Zerr.

206. Proposed by DR. O. E. GLENN, Drury College.

Evaluate $\int_0^1 (1-z^n)^m \frac{\partial}{\partial z}\log(1-z^n x^n)dz$, assuming $-1<x^n<+1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons. W. Va.

$$u=-\int_0^1 \frac{nx^n z^{n-1}(1-z^n)^m}{1-x^n z^n}dz.$$ Let $1-z^n=y$, then we have

$$u=-\int_0^1 \frac{x^n y^m}{1-x^n+x^n y}dy=-\int_0^1 \frac{y^m}{y+a}dy, \text{ where } \frac{1-x^n}{x^n}=a.$$

$$\therefore u=-\int_0^1\left(y^{m-1}-ay^{m-2}+a^2y^{m-3}\ldots\ldots(-1)^{m-1}a^{m-1}+\frac{(-1)^m a^m}{y+a}\right)dy$$

$$=-\left.\frac{y^m}{m}-\frac{ay^{m-1}}{m-1}+\ldots\ldots+(-1)^{m-1}a^{m-1}y+(-1)^m a^m\log(y+a)\right]_0^1$$

$$=-\left[\frac{1}{m}-\frac{a}{m-1}+\ldots\ldots+(-1)^{m-1}a^{m-1}+(-1)^m a^m\log\left(\frac{1+a}{a}\right)\right]$$

$$=-\left[\frac{1}{m}-\frac{(1-x^n)}{x^n(m-1)}+\frac{(1-x^n)^2}{x^{2n}(m-2)}+\ldots\ldots\frac{(-1)^{m-1}(1-x^n)^{m-1}}{x^{(m-1)n}}\right.$$

$$\left.+(-1)^{m+1}\frac{(1-x^n)^m}{x^{mn}}\log(1-x^n)\right].$$

DIOPHANTINE ANALYSIS.

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128. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Required the highest powers of 2, 3, 5, 7, contained in (1000)!

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon. Ill.

$$(1000)!=2^{500}(500)!(1.3.5\ldots\ldots 999)$$
$$(500)!=2^{250}(250)!(1.3.5\ldots\ldots 499)$$

....

Proceeding thus we find the powers required are

$$2^{994},\quad 3^{498},\quad 5^{249},\quad 7^{164}.$$

II. Solution by DR. O. E. GLENN, Drury College.

The theorem covering the general problem is due to Gauss,* and is the following: If p is any prime less than or equal to m, then the highest power of p dividing $m!$ is

$$p^{[m/p]+[m/p^2]+[m/p^3]+\dots} = p^{\sum_{i=1}^{\infty}[m/p^i]}$$

where $[s/t]$ stands for the greatest integer in s/t. Applying this we have $2^{500+250+125+62+31+15+7+3+1}=2^{994}$, and similarly for the others.

Also solved by A. H. Holmes, and G. B. M. Zerr.

GEOMETRY.

263. Proposed by FREDERICK R. HONEY, Trinity College, Hartford, Conn.

Construct a sphere whose surface shall intersect the surface of any four given spheres in great circles.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Let P be the center of a circle intersecting two circles, centers C, C', in the extremities of diameters AB, $A'B'$, respectively. Draw through P a perpendicular to CC', intersecting it in D. Then, if r be the radius of the intersecting circle, we have

$$r^2=PC^2+CA^2=PC'^2+C'A'^2.$$

$$\therefore PD^2+DC^2+CA^2=PD^2+DC'^2+C'A'^2, \text{ and } DC^2-DC'^2=C'A'^2-CA^2.$$

Hence D is a fixed point, and the locus of P is consequently a fixed line. By rotating the figure about CC' we find that the locus of the center of a sphere intersecting two given spheres in great circles is a certain plane.

Constructing these planes for three pairs of the given spheres, each sphere being involved, we get a common point as the center of the required sphere, assuming that the centers of the given spheres are not coplanar.

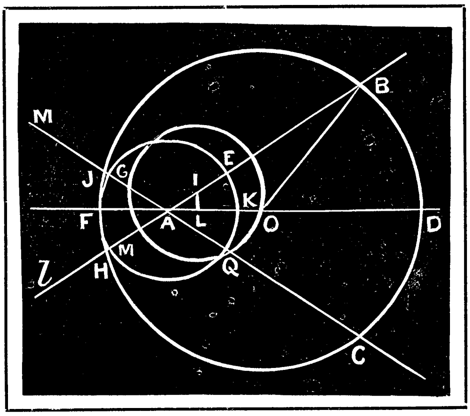
264. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

Let l and m be two straight lines intersecting in A. With A as center and any radius r describe a circle intersecting l and m in E, M and G, Q, respectively; and the bisector of the opposite angles formed by l and m in F and K. With I, the middle point of EA, as center, and radius, r, describe an arc intersecting the bisector of the opposite angles formed by l and m in O. With O as center, and radius $OA+r$ describe circle $FHCDBJF$; F and D the points of intersection of this circle with the bisector of opposite angle; H, B the intersections on l, and J, C on m. What is the ratio of arc HFJ to arc BD?

Disquisitiones Arithmeticae.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let 2θ be the angle formed by l, m. Also let A be the origin. Then $(\frac{1}{2}r\cos\theta, \frac{1}{2}r\sin\theta)$ are the coördinates of I where FD is the axis of abscissas.



$\therefore x^2+y^2-rx\cos\theta-ry\sin\theta=\frac{3}{4}r^2$ is the equation to the circle, center I.

$\therefore y^2-ry\sin\theta=\frac{3}{4}r^2$ or

$y=\frac{1}{2}r\sin\theta \pm \frac{1}{2}\sqrt{(3r^2+r^2\sin^2\theta)}$.

$\therefore OD=OA+r=y+r=r[1+\frac{1}{2}\sin\theta+\frac{1}{2}\sqrt{(3+\sin^2\theta)}]$.

Then if $\angle BOD=\phi$,

$$AO:BO=\sin(\phi-\theta):\sin\theta.$$

$$\therefore \sin(\phi-\theta)=\frac{[\sin\theta+\sqrt{(3+\sin^2\theta)}]\sin\theta}{2+\sin\theta+\sqrt{(3+\sin^2\theta)}}$$

This gives ϕ. Now arcHFJ:arc$BD=2\theta r:\phi DO$.

$\therefore$ arcHFJ:arc$BD=2\theta:[2+\sin\theta+\sqrt{(3+\sin^2\theta)}]\phi$.

265. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Find the Cartesian equation of a curve in a vertical plane such that a particle, sliding down the curve under the force of gravity alone, will require to pass from any point of beginning to the lowest point of the curve, a time proportional to the square of the distance to be traversed along the curve.

Solution by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

With the usual notation, the equation of motion is

$$\frac{ds^2}{dt^2}=2gy \text{......... } (1).$$

Let $t=ks^2$, or $s\sqrt{k}=\sqrt{t}$ ....(2). This gives

$$\frac{d^2s}{dt^2}=\frac{1}{4kt}=\frac{1}{4k^2s^2}=2gy \text{.........(3), or, } s=\frac{1}{2k\sqrt{2g}\sqrt{y}} \text{......... (4).}$$

Differentiating, $ds=-\frac{1}{4k\sqrt{2g}}\cdot\frac{1}{y^{\frac{3}{2}}}$ .... (5). But $dx^2+dy^2=ds^2$ ....(6).

Substituting (5) in (6), and arranging,

$$dx=\frac{\sqrt{(1-16k^2gy^3)}}{4k\sqrt{g}\,y^{\frac{3}{2}}}dy,$$

the differential equation of the curve, cartesian coördinates.

Also solved by G. B. M. Zerr. Professor Greenwood derives the *intrinsic* equation in the simple form $k^2=4s^3g\cos\phi$.

266. Proposed by DR. O. E. GLENN, Drury College.

Given the feet of the three perpendiculars from any point a on the circum-circle to the sides of the triangle are collinear, then if on the three chords $\overline{ab}$, $\overline{ac}$, $\overline{ad}$, as diameters circles be described, the points of intersection of these circles are collinear. [Salmon's *Higher Plane Curves*].

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Let A, B, and C be the points of intersection of the circumferences described on ab, ac, and ad as diameters. Then since the diameters are concurrent at a, and A is common to two circles which have a common chord, $\angle cAa$ and $\angle dAa$ are right angles, and therefore their two non-coincident sides form a straight line through c and d. Likewise B is collinear with b and d, and consequently with c and d. Also C is collinear with b and c since there cannot be more than one perpendicular to the common chord at C. Therefore A, B, and C are collinear.

II. Solution by the PROPOSER.

By inversion with respect to a as a center, ab, ac, ad remain invariant. The given circle inverts into its radical axis with the circle of inversion, and the three circles in question into a triangle in the circle of inversion. The vertices of this triangle lie on a circle passing through a, for perpendiculars from a on the sides are collinear. Hence the invert of a circle through the intersections of the circles in question is a circle through a, and hence the circle through these intersections is a straight line.

Also solved by G. B. M. Zerr.

GROUP THEORY.

10. Proposed by O. E. GLENN, Ph. D.

Find the order of the group of isomorphisms (H) of the group (G) of order p^4 defined by the relations $P_1^{p^2}=P_2^{p^2}=I$, $P_1P_2=P_2P_1$.

Solution by the PROPOSER.

In the general isomorphism represented by

$$*J=\begin{pmatrix} P_1, & P_2 \\ P_1^xP_2^y, & P_1^zP_2^w \end{pmatrix}$$

the operation $P_1^\alpha P_2^\beta$ corresponds to $P_1^{x'}P_2^{y'}$, where

$$\begin{matrix} x'\equiv \alpha x+\beta z \\ y'\equiv \alpha y+\beta w \end{matrix} \pmod{p^2} \;\ldots\ldots\ldots(\mathrm{I}).$$

Hence H is isomorphic with the congruence group defined by (I) and its order h equals the number of sets of values x, y, z, w for which

*Burnside, *Finite Groups*, p. 222.

$$xw - zy \equiv S(\text{mod } p^2)\ ;\ (S \text{ not} \equiv 0)$$

provided only that at least one letter of the set (x, y), and one of (z, w) are prime to p. Consider first the case $zy \equiv 0 \ (\text{mod } p^2)$. This may happen when

1) $z \equiv 0$, with $y \equiv \Phi(p^2) = p(p-1)$ values.
2) $z \equiv 0$, with $y \equiv p^2 - 1 - \Phi(p^2) = p - 1$ values.
3) $z \equiv 0$, $y \equiv 0$.
4) $z \equiv \Phi(p^2) = p(p-1)$ values, with $y \equiv 0$.
5) $z \equiv p^2 - 1 - \Phi(p^2) = p - 1$ values, with $y \equiv 0$.
6) $z \equiv p^2 - 1 - \Phi(p^2) = p - 1$ values, with $y \equiv p^2 - 1 - \Phi(p^2) \equiv p - 1$ values.

Then for

1), x may assume $p^2 - 1$, and w, $p(p-1)$ values.
2), x may assume $p(p-1)$, and w, $p(p-1)$ values.
3), x may assume $p(p-1)$, and w, $p(p-1)$ values.
4), x may assume $p(p-1)$, and w, $p^2 - 1$ values.
5), x may assume $p(p-1)$, and w, $p(p-1)$ values.
6), x may assume $p(p-1)$, and w, $p(p-1)$ values.

Hence there are

$$2(p^2-1)p^2(p-1)^2 + 2p^2(p-1)^3 + 2p^2(p-1)^2 + 2(p^2-1)p^2(p-1)^2 + 2p^2(p-1)^3 + 2p^2(p-1)^4 \ldots\ldots\ldots (\text{II})$$

sets of values with either $xw \equiv 0$ or $zy \equiv 0$, for which S is not$\equiv 0 \ (\text{mod } p^2)$.

Next let two of the parameters x, y, z, w, be divisible by p, without making $xw \equiv 0$ or $zy \equiv 0$. These must be (x and z) or (y and w). Let x and z be multiples of p, $[(p-1)^2$ sets of them$]$. Then to y may be assigned $\Phi(p^2) = p(p-1)$ values, for each one of which there are *$dv(x;\, p^2) = p$ values of w which make $S \equiv 0$ and $p(p-1) - p = p^2 - 2p$ values giving a value of S not$\equiv 0$; and similarly, taking y and w, divisible by p. We thus obtain

$$2p^2(p-2)(p-1)^3 \text{ sets of } x,\ y,\ z,\ w,\ \ldots\ldots\ldots (\text{III}).$$

Next let it be assumed that one and but one parameter x, y, z, w, is divisible by p. Then for *all* the $\Phi(p^2)$ values each, of the remaining three parameters, S is not$\equiv 0$, and since there are four ways of taking one letter divisible by p, and $p-1$ values for that letter, this case gives

$$4p^3(p-1)^4 \text{ additional sets} \ldots\ldots\ldots (\text{IV}).$$

Finally, let all the parameters be prime to p. To each of three of them we may assign any one of $\Phi(p^2) = p(p-1)$ values, and then one value of the remaining one is determined making $S \equiv 0$, so that there are $p(p-1) - 1$ values of

*Mathews, *Theory of Numbers*, p. 11.

this remaining one giving a value of S not$\equiv 0$ (mod p). Thus are obtained $p^3(p-1)^3(p^2-p-1)$ new sets ....(V).

This exhausts the cases to consider. The sum of II, III, IV, and V is h, the order of the automorph of G. This sum is

$$h=p^3(p-1)^2(p+1)(p^2+p-1).$$

MECHANICS.

184. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere, radius a, rests between two parallel thin perfectly rough rods A and B in the same horizontal plane at a distance apart equal to $2c$; the sphere is turned about A until its center is very nearly vertically over A; it is then allowed to fall back. Prove that it will rock between A and B if $10c^2<7a^2$; also, that θ_r, the angle through which it will turn after the rth impact is given by the equation $\cos\theta_r=\dfrac{\sqrt{(a^2-c^2)}}{a}+\dfrac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\dfrac{10c^2}{7a^2}\right)^{2r}$.

I. Solution by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Let ω'=the angular velocity of the sphere about its center at any time t' from the beginning of the first stage of motion; k, k_1 the radii of gyration about the center and a point of the surface, respectively; m=its mass, ω_1, ω_2, etc., the angular velocities about the center just after the first, second, etc., impacts; V'=the linear velocity of the center just before the first impact, $p=(a^2-2c^2)/a$ =the perpendicular from a point of contact upon the direction of V', θ'=the angle the radius makes with the vertical corresponding to ω', and g=the acceleration of gravity. Then for the first impact, the moment of angular momentum is

$$mk_1^2\omega_1=(k^2\omega'+V'p)m\text{ ....(1), or, }\omega_1=\frac{k^2\omega'+V'p}{k_1^2}\text{ ....(2).}$$

For the first stage of motion,

$$mk_1^2\frac{d^2\theta'}{dt'}=mga\sin\theta'\text{ .... (3).}$$

Integrating (3), noticing that initially, $d\theta'/dt'=0$, $\cos\theta'=1$,

$$mk_1^2\frac{d\theta'^2}{dt'^2}=2mga(1-\cos\theta')\text{ .... (4).}$$

When the first stage of motion is completed,

$$\cos\theta'=\frac{\sqrt{(a^2-c^2)}}{a}\text{, and (4) then is, since }k_1^2=\tfrac{7}{5}a^2\text{ ....(5),}$$

$$\omega'^2 = \frac{10g}{7a}\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right) \ldots\ldots\ldots (6).$$

$$\therefore V' = a\omega' = a\sqrt{(2ag)}\sqrt{\frac{5}{7a^2}\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)} \ldots\ldots\ldots (7).$$

(2) then gives, since $k^2 = \frac{2}{5}a^2$,

$$\omega_1^2 = 2ag\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 \div \frac{7}{5}a^2 \ldots\ldots\ldots (8).$$

For rocking, the factor $1-(10c^2/7a^2)$ must be positive, or $10c^2 < 7a^2$.

For the second stage of motion, θ_1, t_1 corresponding to θ', t' in the first,

$$mk_1^2\frac{d^2\theta_1}{dt_1^2} = mga\sin\theta_1 \ldots\ldots\ldots (9).$$

Integrating (9) and noticing that when $\dfrac{d\theta_1}{dt_1} = \omega_1$, $\cos\theta_1 = \dfrac{\sqrt{(a^2-c^2)}}{a}$,

$$mk_1^2\frac{d\theta_1^2}{dt_1^2} = 2mg\left[\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 + \sqrt{(a^2-c^2)} - a\cos\theta_1\right] \ldots\ldots\ldots (10).$$

When the motion ceases for the first time, $d\theta_1/dt_1 = 0$, and then (10) gives

$$\cos\theta_1 = \frac{\sqrt{(a^2-c^2)}}{a} + \left(\frac{a - \sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 \ldots\ldots\ldots (11).$$

For the backward motion, θ'', t'', ω'' corresponding to θ', t', ω', we have

$$mk_1^2\frac{d^2\theta''}{dt''_2} = mga\sin\theta'' \ldots\ldots\ldots (12).$$

Integrating (12), and noticing that, when $\cos\theta'' = \cos\theta_1$, $d\theta''/dt'' = 0$, we have

$$\tfrac{7}{5}ma^2.\omega''^2 = 2mg\left[a\cos\theta'' - \sqrt{(a^2-c^2)} + \left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2\right] \ldots\ldots (13).$$

When

$\cos\theta'' = \dfrac{\sqrt{(a^2-c^2)}}{a}$, (13) gives $\omega''^2 = 2\left(1 - \dfrac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \dfrac{10c^2}{7a^2}\right)^2 g \div \frac{7}{5}a^2 \ldots\ldots (14).$

Now displacing ω_1, ω', V' by ω_2, ω'', V'', we find

$$\omega_2^2 = 2\,ag\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^4 \div \frac{7}{5}a^2 \ldots\ldots\ldots (15);$$

and by a process entirely analogous to that commencing with equation (9), we find

$$\cos\theta_2 = \frac{\sqrt{(a^2-c^2)}}{a} + \left(\frac{a-\sqrt{(a^2-c^2)}}{a}\right)\left(1-\frac{10c^2}{7a^2}\right)^4 \text{.........}(16),$$

and generally, by inference,

$$\cos\theta_r = \frac{\sqrt{(a^2-c^2)}}{a} + \left(\frac{a-\sqrt{(a^2-c^2)}}{a}\right)\left(1-\frac{10c^2}{7a^2}\right)^{2r} \text{.........}(17).$$

I. Solution by the PROPOSER.

Let C be the center of the sphere at the instant of impact, ω_{r-1}, ω_r, the angular velocities of the sphere just before and just after the rth impact, respectively; $\angle ACB = 2\beta$, k^2=radius of gyration, then $(a^2+k^2)\omega_r = (k^2+a^2\cos2\beta)\omega_{r-1}$. Now, $\sin\beta = c/a$, $k^2 = \frac{2}{5}a^2$.

$$\therefore \tfrac{7}{5}a^2\omega_r = a^2\left(\tfrac{7}{5} - \frac{2c^2}{a^2}\right)\omega_{r-1}. \quad \therefore \frac{\omega_r}{\omega_{r-1}} = 1 - \frac{10c^2}{7a^2}.$$

$\therefore 1 - \dfrac{10c^2}{7a^2}$ must always be positive. Let ω=angular velocity just before first impact. By the principle of energy,

$$\tfrac{1}{2}(a^2+k^2)\omega^2 = ga(1-\cos\beta),$$
$$\tfrac{1}{2}(a^2+k^2)\omega_r^2 = ga(\cos\theta_r - \cos\beta).$$

$$\therefore \frac{\omega_r^2}{\omega^2} = \frac{\cos\theta_r - \cos\beta}{1-\cos\beta}. \quad \cos\theta_r = \cos\beta + (1-\cos\beta)\left(\frac{\omega_r}{\omega}\right)^2.$$

$$\cos\beta = \frac{\sqrt{(a^2-c^2)}}{a}, \qquad \frac{\omega_r}{\omega} = \left(1 - \frac{10c^2}{a^2}\right)^r.$$

$$\therefore \cos\theta_r = \frac{\sqrt{(a^2-c^2)}}{a} + \frac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\frac{10c^2}{a^2}\right)^{2r}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

247. Proposed by PROFESSOR G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Find the sum, to n terms, of

$$1 + \frac{n}{2} + \frac{n(n+2)}{2.4} + \frac{n(n+2)(n+4)}{2.4.6} + \text{.........}$$

248. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{6435}{2}.\frac{161280^2}{929569}\left[1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\dots\dots\right]=\pi^{16}$.

249. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $x+y+z=5$, $x^2+y^2=z^2$, $x^3+y^3+z^3=8$.

AVERAGE AND PROBABILITY.

173. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A chord of length c is drawn at random in a given ellipse. What is the average area of the segment cut off by the chord?

CALCULUS.

209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

210. Proposed by EDWIN L. RICH, Schenectady, New York.

Determine a polynomial, $f(x)$, entirely in x and of the seventh degree, so that $f(x)+1$ is divisible by $(x-1)^4$ and $f(x)-1$ by $(x+1)^4$. [Goursat-Hedrick, *A Course in Mathematical Analysis*, p. 32, Ex. 3.]

211. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If $x=v^{1/(v-1)}$, what is the $f(x)$ such that $v=f(x)$?

DIOPHANTINE ANALYSIS.

130. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

In how many ways may a number of which the prime factors are known, be expressed as the sum of two or more consecutive numbers?

131. Proposed by DR. O. E. GLENN, Drury College.

Devise a method of finding the cubic residues of a number, analogous to Gauss's "Method of Exclusion" for quadratic residues.

GEOMETRY.

274. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a straight line AB is placed between two intersecting straight lines MN and PQ and is made to revolve through all possible positions having A always in MN and B always in PQ, what is the locus of any point L in AB or AB produced?

275. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

An hyperbola is drawn touching the axes of an ellipse, and the asymptotes of the hyperbola touch the ellipse. Prove that the center of the hyperbola lies on one of the equal conjugate diameters of the ellipse.

276. Proposed by G. I. HOPKINS, Manchester, N. H.

ABC is an equilateral triangle whose vertices are the centers of circles with radius *AB*, and *H* is the center of the arc *AB*. From *F*, the point of intersection of the circles whose centers are *A* and *C*, a line is drawn through *H* to the circumference *CAN*. Draw *BN*, and prove that the angle *ABN* is an angle of a regular pentagon.

MECHANICS.

186. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

A point *P* keeps at uniform distance from and moves with uniform angular velocity around a point *Q* which is in harmonic motion, making one revolution while *Q* swings to and fro. If *P* is in the line of the path of *Q* and on the same side of the center of that path with *Q* when *Q* is at the extremity of the path, what is the locus of *P*?

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Algebra, 179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^2+qx^2+px+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

Geometry, 267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles is an isothermal system.

Group Theory, 9. Proposed by DR. L. E. DICKSON, The University of Chicago.

Does there exist a triply transitive group on m letters of order $m(m-1)(m-2)$ other than the linear fractional group in the Galois Field of order $p^n=m-1$ and the group 720_3 on ten letters (Cole, *Quarterly Journal*, 1895, p. 44)? This question relates to Problem 99, MONTHLY, March, 1900.

Miscellaneous, 151. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Sum the series $\sum\limits_{r=1}^{r=m} \operatorname{cosec}\left[\frac{2r-1}{4m}\pi+\theta\right]\operatorname{cosec}\left[\frac{2r-1}{4m}\pi-\theta\right]$.

NOTES AND NEWS.

Mr. A. R. Maxson has been appointed tutor in mathematics at Columbia University.

Mr G. H. Light has been appointed instructor in mathematics at Purdue University.

Purdue University has appointed Mr. W. A. Tehring as instructor in pure mathematics.

Mr. Arthur Ranum has been appointed assistant in mathematics at Stanford University.

Miss Florence Hanington has been appointed fellow in mathematics at Bryn Mawr College.

Mr. H. W. Stager and Mr. T. G. Brown are assistants in mathematics at Stanford University.

Mr. W. A. Pickering has received the appointment of professor of applied mathematics at Cardiff.

A technical high school with accommodations for 1200 people will be erected at Newark, N. J.

Miss Louise Duffield Cummings has been appointed scholar in mathematics at Bryn Mawr College.

Mr. Paul N. Peck has been appointed instructor in mathematics in the George Washington University.

Mr. Edwin Haviland, A. M. (Cornell), is a candidate for the doctorate in mathematics at Stanford University.

At Grinnell College, Assistant Professor W. J. Rusk has been promoted to a full professorship in mathematics.

Mr. G. M. Conwell has received the appointment of fellow in mathematics at Princeton University for the year 1905-1906.

At the College of the City of New York Mr. G. C. Daly, Mr. H. W. Powell, and Mr. L. P. Siceloff have been appointed tutors in mathematics.

The Bolyai prize has been awarded to M. Poincaré, in recognition of his researches in mathematics during the past five years. The awarding commission met in Budapest last month.

There will be a Civil Service examination on December 6 to fill a vacancy in the position of computer (male) in the forest service. The salary attached to the position is $1000 per annum.

Professor Arnold Emch, formerly of the University of Colorado, has assumed his new duties as professor of mathematics at his alma mater, the Cantonal College of Solothurn, Switzerland.

On Tuesday, January 9, 1906, Dr. George Bruce Halsted will deliver an address at Ohio State University, under the auspices of the Philosophical Society on "The Non-Euclidean Contribution to Philosophy."

The Academy of Sciences of Berlin held its Leibnitz session on June 29. The sum of six thousand marks was set apart in recognition of the investigations of the late Guido Hauck, but the Steiner prize was not awarded.

Dr. Arthur G. Hall, formerly instructor in mathematics at the University of Michigan, and associate professor of mathematics at the University of Illinois, has been called to the professorship of mathematics at Miami University.

Professor C. J. Keyser is in editorial charge of the mathematical department of the Encyclopedia Americana, about to be issued by the *Scientific American.* Special attention has been given to this department, and over forty articles have been written for the Encyclopedia by American mathematicians on their several special fields of investigation.

The following have been elected members of the American Mathematical Society: Lieutenant Colonel C. P. Echols, U. S. Military Academy; Professor G. B. Guccia, University of Palermo; Professor H. B. Evans, University of Pennsylvania; Dr. A. M. Hiltebeitel, Princeton University; Dr. J. M. Poor, Dartmouth College; Professor J. E. Williams, Virginia Polytechnic Institute.

The London Mathematical Society held its annual general meeting at the Registered Office of the Society, 22 Albemarle Street, London, on November 9th. The following officers were elected for the ensuing year: President, A. R. Forsyth, Sc. D., F. R. S; Vice-Presidents, W. F. Burnside, Sc. D., F. R. S., Sir William Niven, K. C. B., F. R. S.; Treasurer, J. Larmor, D. Sc., F. R. S.; Secretaries, A. E. H. Love, D. Sc., F. R. S., and J. H. Grace, M. A.

The American Mathematical Society held its October meeting at Columbia University on Saturday, October 28th. The program issued for the meeting announced the following papers: "On the Cayley-Veronese Class of Configurations," by Professor W. B. Carver; "Multiple Improper Integrals," by Professor James Pierpont; "Poncelet Quadrilaterals on a Curve of the Third Order and a Conic," by Professor H. S. White; "On the Geodesics Passing through a Given Point of a Surface," by Dr. Edward Kasner.

The November number of the *Bulletin of the American Mathematical Society* contains the following papers: "A Set of Generators for Ternary Linear Groups," by Ida May Schottenfels; "Note on the Structure of Hyper-Complex Number Systems," by Saul Epsteen; "A Geometric Property of the Trajectories of Dynamics," by Edward Kasner; "On the Possible Numbers of Operators of

Order 2 in a Group of Order 2^m," by G. A. Miller; "On the Arithmetic Nature of the Coefficients in Groups of Finite Monomial Linear Substitutions," by W. A. Manning.

The fifth annual meeting of the Central Association of Science and Mathematics Teachers will be held in Chicago (153 LaSalle Street) on November 30, December 1 and 2. The following addresses will be made before the mathematics section: "The Straight Line in Geometry," by J. W. Withers, Teachers College, St. Louis; "Interest and Progress in the Teaching of Mathematics," by N. J. Lennes, Wendell Phillips High School, Chicago; "Aims in Teaching Algebra," by Professor Robert J. Aley, Indiana University; "Some Thoughts on the Teaching of Geometry," by C. A. Patterson.

Rev. J Edward Kirbye, D. D., was inaugurated as the fourth president of Drury College on November 9th. The installation ceremonies were signalized by a special reunion of the alumni of the college and its friends. "Democracy and Higher Education" was the subject of President Kirbye's inaugural address. Other educators who delivered addresses were Rev. Stephen M. Newman of Washington, D. C., Chancellor W. S. Chaplin of Washington University, Professor John Picard of the University of Missouri, Professor George E. Comstock of the University of Wisconsin, Father Roger of St. Louis University.

At the last commencement at Grinnell College, Professor Samuel J. Buck retired from active service and was made professor emeritus after a continuous service of forty-one years at that institution. He graduated from Oberlin College in 1858. In 1862, he graduated from Oberlin Theological Seminary, entered the University in 1864, was made principal of the Academy in Grinnell College in 1864, and made professor of mathematics and physics in the college in 1869. In 1893 he was elected professor of mathematics and astronomy at the same college, which position he has held up to the present time. July fourth, 1905, Professor Buck was seventy years old. He is in excellent health and leaves the honored position which he has made and brought to high college rank after more than two scores of years of service, appreciation for which was shown in marked way by the friends of the college at the time of his retirement.

Princeton University has, during the past year, adopted and put into practice the preceptorial system of instruction, largely increasing her staff of professors in all departments. As a result, the following changes have gone into effect in the mathematical department this Fall: Mr. J. H. Jeans, of Trinity College, Cambridge has been appointed professor of applied mathematics to fill the chair vacated by Prof. E. O. Lovett on his being transferred to the department of astronomy. Dr. L. P. Eisenhart and Dr. William Gillespie have been promoted to preceptorships and the following have been called from other institutions to preceptorships: Dr. G. A. Bliss from the University of Missouri, Dr. Oswald Veblen from the University of Chicago, Professor John W. Young from the Northwestern University. The newly appointed instructors are: C. R. MacInnes from the Johns Hopkins University, A. L. Underhill and E. B. Morrow.

A curious poetical tribute to Archimedes, composed by a French mathematician, is recalled by *The Academy* (London). The first calculation of the numerical value of π is ascribed to Archimedes, and, carried out to thirty decimal places, this value is 3.141592653589793238462643383279. Each of the thirty-one words in the quatrain, in order, contains the number of letters given by the corresponding figure in the numerical value of π, as follows:

3 1 4 1 5 9 2 6 5 3 5
Que j' aime á faire apprendre un nombre utile aux sages
8 9 7 9
Immortel Archimede, artiste ingénieur!
3 2 3 8 4 6 2 6
Qui de ton jugement peut priser la valeur
4 3 3 8 3 2 7 9
Pour moi ton problème eut de pareils avantages.

The *Frankfurter Zeitung* adds the following similar "effort" from a German "poet and geometrician:"

3 1 4 1 5 9 2 6 5
Dir o Held, o alter Philosoph, Du Riesen-Genie!
3 5 8 9 7 9 3 2 3 8
Wie viele Tausende bewundern Geister himmlisch wie Du und göttlich!
4 6 2 6 4 3 3 8 3 2 7 9
Noch reiner in Aeonen wird das uns strahlen wie im lichten Morgenrot!

The translation of the French is: "How I love to teach a number useful to the wise immortal Archimedes, artist—engineer! Who can appraise the worth of thy judgment? For me thy problem has equal advantages." The German runs thus: "To thee, O hero, O old philosopher, O giant genius! How many thousand souls wonder, heavenly as thou and divine! Yet clearer in the ages will that stream on us than in the luminous dawn." Neither of these can be said to be very clear, but considering the limitations of their composition they are certainly remarkable, gravely comments the *Literary Digest*. Meanwhile, will some one give us an English version of π?

BOOKS AND PERIODICALS.

Advanced Algebra. By Herbert E. Hawkes, Ph. D., Assistant Professor of Mathematics in Yale University. Cloth, 8vo., XIV+285 pages. List price, $1.40; mailing price, $1.50. Ginn & Co., Boston, New York, Chicago, London.

An algebra preëminently suited for short college courses should be comprehensive enough to set forth amply all the parts of the science that are recognized as essentials, and it should be terse enough to be covered in a college term without skipping chapters and parts of chapters, and thus interfering with the continuity of the student's development. *Advanced Algebra* is a book of decided merit when judged from this standpoint. The parts of the science usually taught in college freshman algebra courses are presented in less than three hundred pages, and the treatment is, on the whole, exceptionally clear and forceful. The text is full of the laboratory spirit; graphical representations, and graphical methods of solution of equations form an integral part of Professor Hawkes' work, and they are

introduced early in the text. Necessarily the brevity of treatment has curtailed fundamental and even indispensable general theory, but the book is primarily aimed to "present in concise but clear form the portions of algebra that are required for entrance to the most exacting universities and technical schools," and it will supply a definite want in this field. G.

The Essentials of Algebra. By Robert J. Aley, Ph. D., and David A. Rothrock, Ph. D., Professors of Mathematics at Indiana University. Chapter XX, Logarithms; 15 pages. Silver, Burdett & Co., New York, Boston, Chicago.

This chapter is bound with the new edition of this successful and in many respects epoch-marking secondary school algebra, at the request of a number of friends of the book who wish to teach the theory of logarithms in their high school courses. The chapter is also to be had in pamphlet form. The main theorems of the subject are developed and a five place table for argument one to one hundred given. The principles find application in a treatment of compound interest and annuities. G.

Elements of Descriptive Geometry. By O. E. Randall, Ph. D,, Professor of Mechanical Drawing, Brown University. 8vo. Semi-flexible cloth. IV+209 pages. List price, $2.00; mailing price, $2.10. Ginn & Co., Boston, New York, Chicago, London.

This work is a text book for colleges and engineering schools. The aim otfhe treatise is to make a clear presentation of the theory of projection and to show its application as a medium of expression. By the discussion and proof of a great variety of problems the author aims to enable the student to make a ready and intelligent use of this medium in the representation of all forms of magnitudes. Since by far the greater part of practical drafting is done from the standpoint of the third quadrant, the principles of descriptive geometry are presented from the standpoint of the same quadrant. It seems to us that Professor Randall's treatise is one of the very best of a beautiful line of texts in this subject which has appeared in the past few years. The principles of the subject are here classified and pedagogically arranged, and are illustrated by numerous figures and drawings. In fact, this is the first treatise on the subject that has come to our notice which makes the illustrative drawing do its full part. Some special features will commend themselves to teachers of mechanical and descriptive drawing:

Free use is made of profile and other supplementary planes of projection. Isometric projection and other forms of one-plane projection are treated as applications of descriptive geometry. A method of locating given parts is incorporated. As a result the work of the drawing and recitation room may be easily and definitely assigned. G.

Transactions of the American Mathematical Society. Published quarterly for the Society by the Macmillan company.

The closing (October) number of Volume 6 of the *Transactions of the American Mathematical Society* contains the following papers: Maurice Fréchet, "Sur l'écart de deux courbes et sur les courbes limites:" John Eisland, "On a Certain System of Conjugate Lines on a Surface Connected with Euler's Transformation;" L. P. Eisenhart, "Surfaces of constant Curvature and their Transformations;" N. J. Lennes, "Volumes and Areas;" E. O. Lovett, "On a Problem Including that of Several Bodies and Admitting of an Additional Integral;" F. R. Sharpe, "On the Stability of the Motion of a Viscous Liquid;" A. Loewy, "Ueber die Vollständig Reducibeln Gruppen, die zu einer Gruppe Linearer Homogener Substitutionen Gehören;" W. B. Carver, "On the Cayley-Veronese Class of Configurations." This number also contains; Notes and Errata, Volumes 5, 6; Table of Contents, Volume 6.

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EXPRESSIONS FOR THE ELEMENTS OF A DETERMINANT IN TERMS OF THE MINORS OF A GIVEN ORDER. GENERALIZATION OF A THEOREM DUE TO STUDNICKA.

By L. E. DICKSON.

I. It is desired to express the n^2 elements a_{ij} of a determinant $D\equiv|a_{ij}|$ in terms of the $C^2{}_{n,m}$ minors of order m. Arrange these minors in a determinant $\triangle$ so that there shall appear in one row [column] all the $C_{n,m}$ minors built from the elements of the same m rows [columns] of D; then $\triangle$ is called the mth compound of D. It is well known that

$$D^{C_{n-1,\, m-1}}=\triangle. \tag{1}$$

For the applications in view it suffices to assume that $\triangle\neq 0$, whence $D\neq 0$.

One symmetrical method of treating the problem consists in determining all the minors M_{m-1} of order $m-1$, then all the minors M_{n-2}, ..., and finally the a_{ij}. We first determine D from (1) by a root extraction.* By Studnicka's

*We can however find D by the extraction of a dth root, where d is the greatest common divisor of m and n, since D^d equals a sum of products of the M_m. For instance, if $d=1$, this follows from Laplace's expansion of D in the form $\Sigma\neq M'_m\, M''_m \,........\, M_m^{(n/m)}$, where M'_m is formed from the first m rows of D, M''_m from the second m rows, etc. Again, if $d=2$ take $n=6$, $m=4$ for brevity of writing; then

theorem, a very special case of formula (6) proved below, $M_{m-1}^{n-m}D$ can be expressed rationally in terms of the M_m, so that any minor M_{m-1} can be found by root extraction.

Such a method may properly be rejected since it introduces an array of radicals of various orders. It is in fact possible to express the a_{ij} in terms of the minors M_m so that the *only* irrationality entering is the mth root of a certain rational function of the M_m. It is evident *a priori* that the a_{ij} are at least m valued functions of the M_m. For, if the a'_{ij} give one set of solutions and if ε is an mth root of unity, then the products $\varepsilon a'_{ij}=a''_{ij}$ give another set of solutions, since $|\,a''_{ij}\,|_m=\varepsilon^m\,|\,a'_{ij}\,|_m$. We shall henceforth demand solutions a_{ij} which are exactly† m-valued functions of the M_m.

2. I give first a very elementary method of solution. For general m I rely upon the argument now given for the simplest case $m=n-1$. Let A_{ij} be the co-factor (minor of order $n-1$ with prefixed sign) of a_{ij} in D. Then $\Delta=|\,A_{ij}\,|=D^{n-1}$, so that D is the mth root of a rational function of the M_m. It is well known that the determinant of order $n-1$, obtained from Δ by deleting the ith row and jth column, equals $D^{n-2}a_{ij}$, apart from sign. Hence the a_{ij} are rational functions of the M_m and of the mth root of a certain rational function of the M_m.

3. By way of illustration, consider the case $m=2$ (n arbitrary), so that the minors M_2 of order 2 are given. From formulae like

$$(2)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^2 = \begin{vmatrix} A_{33} & A_{32} & A_{31} \\ A_{23} & A_{22} & A_{21} \\ A_{13} & A_{12} & A_{11} \end{vmatrix},$$

where A_{ij} is the minor of a_{ij} in $|\,a_{ij}\,|$, we can test whether or not a given M_3 vanishes. Since the M_3 built from the first three rows of D do not all vanish, we assume that, for example, the determinant in the left member of (2) is not

$$D^2=\begin{vmatrix} D & O \\ O & D \end{vmatrix}=\left|\begin{array}{llll} 11 & 12 \cdots\cdots 16 & 0 & 0 \cdots\cdots 0 \\ 21 & 22 \cdots\cdots 26 & 0 & 0 \cdots\cdots 0 \\ 31 & 32 \cdots\cdots 36 & 0 & 0 \cdots\cdots 0 \\ 41 & 42 \cdots\cdots 46 & 0 & 0 \cdots\cdots 0 \\ \hline 51 & 52 \cdots\cdots 56 & 51 & 52 \cdots\cdots 56 \\ 61 & 62 \cdots\cdots 66 & 61 & 62 \cdots\cdots 66 \\ 11 & 12 \cdots\cdots 16 & 11 & 12 \cdots\cdots 16 \\ 21 & 22 \cdots\cdots 26 & 21 & 22 \cdots\cdots 26 \\ \hline 0 & 0 \cdots\cdots 0 & 31 & 32 \cdots\cdots 36 \\ 0 & 0 \cdots\cdots 0 & 41 & 42 \cdots\cdots 46 \\ 0 & 0 \cdots\cdots 0 & 51 & 52 \cdots\cdots 56 \\ 0 & 0 \cdots\cdots 0 & 61 & 62 \cdots\cdots 66 \end{array}\right|,$$

as shown by adding the 1st row to the 7th, 2d to 8th, 11th to 5th, and 12th to 6th. By Laplace's expansion, the last is expressed in terms of the M_4. It is readily shown that D^d is the lowest power of D which is rational in the M_m.

†A direct proof that the solutions are expressible as m-valued functions is given in *Linear Groups*, p. 146, §154. I did not consider there the present problem of exhibiting the solutions.

zero. Then by §2, the nine elements $a_{11}, \ldots\ldots, a_{33}$ can be readily expressed rationally in terms of the M_2 and the square root of the second determinant (2).

Not all the minors A_{33}, A_{32}, A_{31} vanish, say $A_{32} \neq 0$. Having the minors $a_{11}a_{2i}-a_{21}a_{1i}$ and $a_{13}a_{2i}-a_{23}a_{1i}$, we get a_{2i} and a_{1i} since the determinant of their coefficients is $-A_{32}$. In this way we can determine $a_{1i}, a_{2i}, a_{3i}, a_{i1}, a_{i2}, a_{i3}$ $(i=4, \ldots\ldots, n)$. Of the elements previously found let $a_{13} \neq 0$, for example. We may now find a_{rs} $(r>3, s>3)$ from $a_{13}a_{rs}-a_{1s}a_{r3}$.

4. As a further illustration, consider the case $m=3$. Of the determinants of order 4 built from the first four rows of D, suppose that

$$(3) \qquad \begin{vmatrix} 12 & 13 & 14 & 15 \\ \ldots & \ldots & \ldots & \ldots \\ 42 & 43 & 44 & 45 \end{vmatrix} \neq 0.$$

By §2, the sixteen elements of (3) are readily expressed as rational functions of the minors M_3 and the cube root of a rational function of the M_3. We next determine a_{1r}, a_{2r}, a_{3r} $(r=1, 6, 7, \ldots\ldots, n)$. Consider the six M_3 :

$$a_{3r}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} - a_{2r}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{3i} & a_{3j} \end{vmatrix} + a_{1r}\begin{vmatrix} a_{2i} & a_{2j} \\ a_{3i} & a_{3j} \end{vmatrix} \qquad (i, j=2, 3, 4, 5;\ i<j).$$

From three of these M_3 we can find $a_{3r}, -a_{2r}, a_{1r}$ if* the determinant of their coefficients is $\neq 0$. For $i, j=2, 3;\ 2, 4;\ 3, 4$, this determinant is

$$\begin{vmatrix} \begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 32 & 33 \end{vmatrix} & \begin{vmatrix} 22 & 23 \\ 32 & 33 \end{vmatrix} \\ \begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} & \begin{vmatrix} 12 & 14 \\ 32 & 34 \end{vmatrix} & \begin{vmatrix} 22 & 24 \\ 32 & 34 \end{vmatrix} \\ \begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} & \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 12 & 13 & 14 \\ 22 & 23 & 24 \\ 32 & 33 & 34 \end{vmatrix}^2.$$

Indeed, by interchanging rows and columns in the first, we have a special case of (1). For $i, j=2, 3;\ 2, 5;\ 3, 5$, the determinant equals

$$\begin{vmatrix} 12 & 13 & 15 \\ 22 & 23 & 25 \\ 32 & 33 & 35 \end{vmatrix}^2,$$

etc. But not all minors of order 3 built from the first three rows of (3) vanish. Thus a_{1r}, a_{2r}, a_{3r} may be considered to be found. Of the minors from the first two rows of (3), let $a_{13}a_{24}-a_{14}a_{23} \neq 0$. We may thus get a_{4r} from

*This determinant vanishes if $i, j=2, 3;\ 2, 4;\ 2, 5$.

$$\begin{vmatrix} 1r & 13 & 14 \\ 2r & 23 & 24 \\ 4r & 43 & 44 \end{vmatrix}.$$

Similarly, we can find every a_{s2}, a_{s3}, a_{s4}, a_{s5}. Finally, from

$$\begin{vmatrix} 1l & 13 & 14 \\ 2l & 23 & 24 \\ kl & k3 & k4 \end{vmatrix}$$

we determine a_{kl} (k any $\neq 1, 2, 3, 4$; l any $\neq 2, 3, 4, 5$).

5. The following method of solution for the important case $m=2$ offers greater symmetry than that of §3. We show that, if $t, i_1, i_2, \ldots\ldots, i_{n-2}$ are any positive integers $\overline{\gtrless} n$, $a_{ti_1}, a_{ti_2}, \ldots\ldots, a_{ti_{n-2}}D$ equals a certain determinant of order $n-1$ whose elements are all minors of degree 2 of $D \equiv |a_{ij}|$. Let j_1 be any integer $\neq i_1$, of the set $1, \ldots\ldots, n$; let j_2 be any one $\neq i_2$ or j_1; j_3 any $\neq i_3, j_1, j_2$; $\ldots\ldots$; j_{n-2} any $\neq i_{n-2}, j_1, j_2, \ldots\ldots, j_{n-1}$; j_{n-1} and j_n the remaining two. Set $i_{n-1} = j_n$. Call C_i the ith column in D. Subtract $a_{tj_1}C_{i_1}$ from $a_{ti_1}C_{j_1}$, $a_{tj_2}C_{i_2}$ from $a_{ti_2}C_{j_2}$, $\ldots\ldots$, $a_{tj_{n-1}}C_{i_{n-1}}$ from $a_{ti_{n-1}}C_{j_{n-1}}$. Hence $a_{ti_1}\ldots\ldots a_{ti_{n-1}}D$ equals a determinant the elements of whose j_nth column are the same as in D, while the element lying in the j_sth column ($s<n$) and rth row is

$$a_{ti_s}a_{rj_s} - a_{tj_s}a_{ri_s} \equiv T_{rj_s}.$$

Hence in the tth row all the elements are zero except that in the j_nth column. Thus the new determinant is a_{tj_n} times its co-factor. Hence

$$(4) \quad a_{ti_1}\ldots a_{ti_{n-2}}D = (-1)^{t+j_n}\,|\,T_{rj_s}\,| \quad (s=1, \ldots, n-1;\ r=1, \ldots, t-1, t+1, \ldots, n),$$

the sequence of columns being that of the order of magnitude of $j_1, \ldots\ldots, j_{n-1}$. By interchanging the rows and columns of D we obtain similarly

$$(5) \qquad a_{i_1t}\ldots a_{i_{n-2}}D = \text{rational functions of minors of degree 2.}$$

Giving fixed values to $i_2, \ldots, i_{n-2}$, and varying i_1, we conclude that the ratio of any two elements a of the same row (or column) is a known rational function of the minors of degree 2. Thus if $a_{t\sigma} \neq 0$, $a_{tl} = R_l = a_{t\sigma}$, $a_{k\sigma} = R_k' a_{t\sigma}$, for every l, k. To prove that $a^2{}_{t\sigma}$ is a rational function of the minors of degree 2, consider any non-vanishing minor D_4 of degree 4 of D. By Laplace's expansion, D_4 is a rational function of the minors of degree 2. Applying (1) for $n=4$, $i_1 = i_2 = \sigma$, we conclude the same for $a^2{}_{t\sigma}D_4$. Hence $a^2{}_{t\sigma} = R$, where R is a rational function of the minors of degree 2. Now the minor

$$\begin{vmatrix} a_{t\sigma} & a_{tl} \\ a_{k\sigma} & a_{kl} \end{vmatrix} = a_{t\sigma}a_{kl} - R_lR'_k a^2{}_{t\sigma} = R(a_{kl}/a_{t\sigma} - R_lR'_k).$$

Hence $a_{kl} \div a_{t\sigma}$ is a rational function of the minors of degree 2. *Every element a is a rational function of $\sqrt{R}$ and the minors.*

6. In conclusion I establish a general theorem on determinants which includes (4) as one special case, and Studnicka's theorem (cited in §1) as another very special case. Let $r_1, r_2, \ldots, r_{m-1}$ be any distinct integers of the set $1, \ldots, n$; likewise for $i_{11}, i_{12}, \ldots, i_{1\,m-1}, j_1$. Let r be any integers $\gtreqless n$. In

$$M_{rj_1} \equiv \begin{vmatrix} a_{ri_{11}} & a_{ri_{12}} \cdots\cdots a_{ri_{1m-1}} & a_{rj_1} \\ a_{r_1 i_{11}} & a_{r_1 i_{12}} \cdots\cdots a_{r_1 i_{1m-1}} & a_{r_1 j_1} \\ \cdots & \cdots & \cdots \\ a_{r_{m-1} i_{11}} & a_{r_{m-1} i_{12}} \cdots a_{r_{m-1} i_{1m-1}} & a_{r_{m-1} j_1} \end{vmatrix}$$

call $A_{ri_{11}}, \ldots, A_{rj_1}$ the co-factors of the elements of the first row. Call C_i the ith column in $D \equiv |a_{ij}|$. Multiply $C_{i_{11}}$ by $A_{ri_{11}}$, $C_{i_{12}}$ by $A_{ri_{12}}, \ldots, C_{i_{1m-1}}$ by $A_{ri_{1m-1}}$, and add the products to $C_{j_1} \times A_{rj_1}$. Hence $A_{rj_1}D$ equals a determinant whose elements outside column C_{j_1} are the same as in D, while the element lying in the j_1th column and rth row is M_{rj_1}, whence those in the rows $r_1, \ldots, r_{m-1}$ are zero. Next, let $i_{21}, i_{22}, \ldots, i_{2m-1}, j_2$ be any distinct integers other than j_1 of the set $1, \ldots, n$. In

$$\mathrm{M}_{rj_2} \equiv \begin{vmatrix} a_{ri_{21}} & a_{ri_{22}} & \cdots\cdots a_{ri_{2m-1}} & a_{rj_2} \\ a_{r_1 i_{21}} & a_{r_1 i_{22}} & \cdots\cdots a_{r_1 i_{2m-1}} & a_{r_1 j_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r_{m-1} i_{21}} & a_{r_{m-1} i_{22}} & \cdots\cdots a_{r_{m-1} i_{2m-1}} & a_{r_{m-1} j_2} \end{vmatrix}$$

call $A_{ri_{21}}, \ldots, A_{rj_2}$ the co-factors of the elements of the first row. Add $C_{i_{21}}A_{ri_{21}}, \ldots, C_{i_{2m-1}}A_{ri_{2m-1}}$ to $C_{j_2} A_{rj_2}$. Hence $A_{rj_1}A_{rj_2}D$ equals a determinant whose elements outside columns C_{j_1} C_{j_2} are the same as in D, while the elements in columns j_1 and j_2 and rth row are M_{rj_1} and M_{rj_2}, respectively. For the third step of the argument, $i_{31}, \ldots, i_{3m-1}, j_3$ are any distinct integers other than j_1, j_2 of the set $1, \ldots, n$. For the $(n-m+1)$th step, $i_{n-m+1}, \ldots, i_{n-m+1\;m-1}$, j_{n-m+1} are any distinct integers other than $j_1, j_2, \ldots, j_{n-m}$ of the set $1, \ldots, n$. After this final step, $A_{rj_1} A_{rj_2}, \ldots, A_{rj_{n-m+1}}D$ equals a determinant whose elements outside columns $C_{j_1}, \ldots, C_{j_{n-m+1}}$ are the same as in D, while the elements in these columns and the rth row are $M_{rj_1}, \ldots, M_{rj_{n-m+1}}$, respectively. In particular, the latter elements are obviously zero for $r = r_1, \ldots, r_{m-1}$. By Laplace's development the determinant is, apart from sign, $A_{rj_{n-m+1}} \times$ co-factor. The sign is determined by the fact that our co-factor is gotten by deleting the rows $r_1, \ldots, r_{m-1}$ and the columns $i_{n-m+1\;1}, \ldots, i_{n-m+1\;m-1}$. Thus

$$(6) \qquad A_{rj_1} A_{rj_2}, \ldots, A_{rj_{n-m}} D = \pm \; | M_{rj_s} |$$

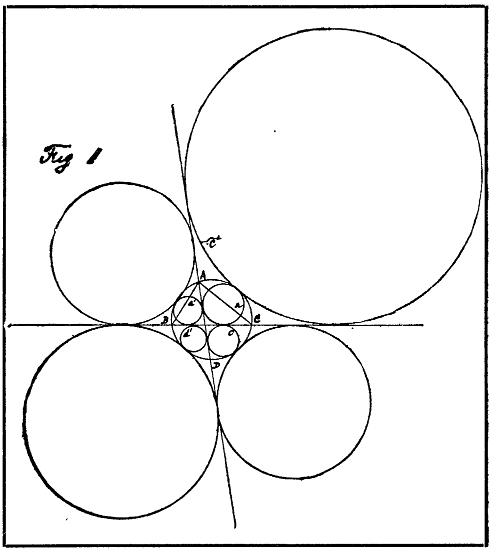
$(s=1, \ldots, n-m+1;\; r=1, \ldots, n;\; r \neq r_1, \ldots, r_{m-1})$, where the diagonal term of the determinant A_{rj_s} is $a_{r_1 i_{s1}} a_{r_2 i_{s2}}, \ldots, a_{r_{m-1} i_{sm-1}}$.

THE UNIVERSITY OF CHICAGO, *November, 1905.*

A NEW GEOMETRICAL PROPOSITION.

By Y. SAWAYAMA, Instructor in The Central Militarv School for Boys, Tokyo, Japan.

General enunciation.—Describe eight circles, each of which touches a circle and any two secants of it; next construct a triangle by joining any three of the points of intersection of the latter circle and the two secants; then the chords of contact and the line of centers of two of the eight circles taken appropriately in pairs are concurrent, the point of concurrency being equidistant from the three sides of the triangle.

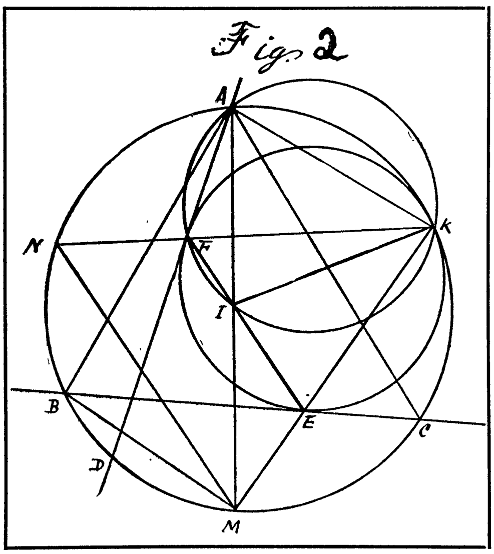


Particular enunciation. — Call the circle, ABC, the eight circles a, a', b, b', c, c', d, d', and the two secants AD, BC (Fig. 1). Then

1. The line of centers of the two circles a and a', and their two chords of contact pass through the center of the inscribed circle of the triangle ABC.

2. The line of centers of the two circles b and b', and their two chords of contact pass through the center of the circle escribed to the side BC of the triangle ABC.

3. The line of centers of the two circles c and c', and their two chords of contact pass through the center of the circle escribed to the side CA of the triangle ABC.

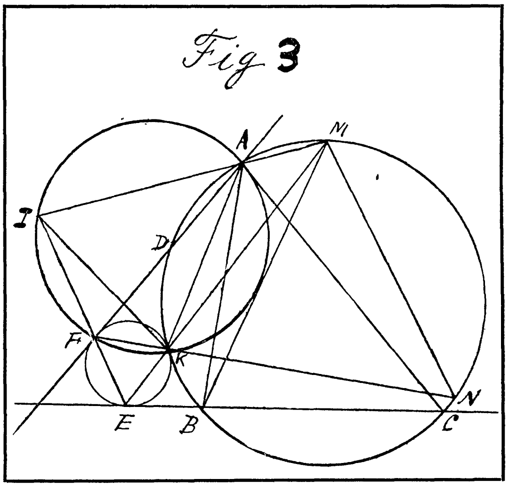


4. The line of centers of the two circles d and d', and their two chords of contact pass through the center of the circle escribed to the side AB of the triangle ABC.

Demonstration.—Let K, E, and F be the three points of contact where any one of the eight circles touches the circle ABC and the two secants BC, AD (Figs. 2 and 3). Then as is well known, we have two properties:

1. The line KE passes through the middle point M of the arc BC, so also the line KF through the middle point N of the arc AD.

2. The two straight lines EF and MN are parallel.



From the last property we have $\angle EFK = \angle MNK$, and as the four points M, N, A, and K lie on the circle ABC, $\angle MNK = \angle MAK$ (Fig. 2), or $=$its supplementary angle (Fig. 3).

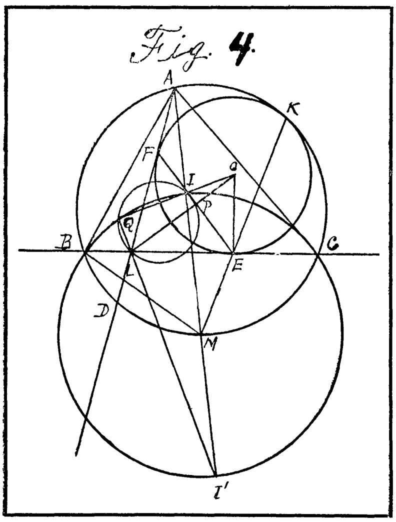
$\therefore \angle EFK = \angle MAK$ or$=$its supplementary angle. Hence the four points A, K, F, and I (which is the intersection of EF and AM) are concyclic.

$\therefore \angle AIK = \angle AFK$ (Figs. 2 and 3), or$=$its supplementary angle (figure omitted). Now the line AF touches the circle EFK, hence $\angle AFK$ or its supplementary angle$=FEK$. $\therefore \angle AIK = \angle FEK$.

$\therefore \triangle MKI$ and $\triangle MIE$ are mutually equiangular, from which we can deduce that the circle passing through the three points I, E, and K touches the line MI, hence $MI^2 = MK.ME$. On the other hand, since arcMB=arcMC, we have $\angle MKB = \angle MBE$ (in the figures we omit the line BK).

$\therefore$ The circle passing through the three points B, E, and K touches the line MB, hence we have $MK.ME = MB^2$, or $MI^2 = MB^2$, $MI = MB$.

This shows that I is equidistant from the three sides of the triangle ABC.



Next, call O the center of the circle EFK, L the intersection of BC and AD, P the intersection of EF and OL, Q another intersection of OI and the circle BIC, and draw the straight line passing through Q and L (Fig. 4).

Then the square of the tangent drawn from the center of the circle BIC to the circle EFK is equal to the rectangle $MK.ME$, or the square of the radius of the circle BIC.

This last fact shows that the two circles EFK and BIC are orthogonal, and the radius OE of the circle EFK is equal to the tangent drawn from the point O to the circle BIC, and that $OE^2 = OQ.OI$.

Again, in the right-angled triangle OLE, $OE^2 = OL.OP$. $\therefore OQ.OI = OL.OP$.

$\therefore$ The four points I, P, L, and Q are concyclic.

Noting the angle IPL is a right angle and the $\angle IQL$ and $\angle IPL$ are equal or supplementary, we have $\angle IQL$=right angle.

$\therefore$ The line QL passes through a point I' which is equidistant from the three sides of the triangle ABC (I' is, in fact, the other extremity of the diameter passing through I of the circle BIC). And the line OI is perpendicular to a fixed line $I'L$ passing through the fixed point Q on it.

Similarly, the center of another circle whose chord of contact passes through the point I, and the point I are on the straight line which is perpendicular to $I'L$ and passes through the fixed point Q on it. Therefore the centers of the two circles whose chords of contact pass through I and the point I are collinear.

Corollary I. In the preceding figures, the point of intersection (other than I) of the circle BIC and the line EF is one of the points which are equidistant from the three sides of the triangle DBC.

One of the two intersections of the line EF and the circle whose center is N and whose radius is NA is equidistant from the three sides of the triangle BAD, and the other from the three sides of the triangle CAD.

Corollary II. In Fig. 1 the points of contact of the eight circles with either of the two secants are in involution, the center of which is the intersection L of the two secants, and then its constant is equal to the power of the point L with respect to the circle ABC.

Demonstration. Take L, the intersection of two secants AD and BC as the center of inversion, and the power of L with respect to the circle ABC for the constant of inversion. In the figures 2, and 3, call E', F', and K', the inverses of the three points E, F, and K, respectively. Then as the two secants EC and AD, and the circle ABC are their own inverses, the circle $E'F'K'$ which is the inverse of the circle EFK, touches the secants BC and AD and the circle ABC.

Corollary III. When AD and BC are perpendicular, the centers of the four circles a, b, a', b' are concyclic; so also those of a, b, c, d; those of a', b', c', d', and those of c, d, c', d'.

DEPARTMENTS.

*DISCUSSION.

THE TANGENT NORMALS TO A LIMACON.

By F. H. SAFFORD, Ph. D., The University of Pennsylvania.

This problem was suggested by Dr. Glenn at the end of the solution of Problem 254, August-September, 1905.

Let the required line be $2ax+2by+c=0$, which is to be both tangent and normal to

$$(x^2+y^2+cx)^2=\frac{c^2}{e^2}(x^2+y^2).$$

Two special cases may be disposed of first: one is that of tangent lines parallel to the Y-axis, the other the case of tangents passing through the origin.

The latter is excluded from the general case by the particular value chosen for the constant term in the straight line, but it may be shown that the only line which satisfies the problem in this case is $y=0$ for $e=1$, *i. e.* the axis of the cardioid.

The former case leads to a solution when $e=\frac{3+\sqrt{5}}{2}$, the point of tangency being at the extremity of the loop, while the line is normal at both of the remaining intersections. The general solution now proceeds under the assumptions $b\neq 0$, $c\neq 0$, and is entirely analogous to the solution previously given, p. 157, for the case in which $e=1$.

Transform the line and limaçon by the inversion

$$x=\frac{cx'}{x'^2+y'^2},\ y=\frac{cy'}{x'^2+y'^2},$$

obtaining $x^2+y^2+2ax+2by=0$, $x^2(1-e^2)+y^2-2e^2x-e^2=0$.

Since angles are unchanged by inversion the problem is now to determine a and b so that the circle shall be both tangent and normal to the conic. The two curves have a point of orthogonal intersection when $x=-1$ or $1-2a$, but the first value leads to $x^2+y^2=0$, which is excluded. When $x=1-2a$, in which $a\neq 1$ for the same reason, then y and b may be found in terms of a as a parameter.

It will simplify later work to write $a=1+h$ $(h\neq 0)$, $e^2=1+d$, so that the point in question is

*The department *Discussion* will be devoted primarily to generalizations and extensions of problems solved in our columns. Some important researches have resulted from such generalizations in the past and it is believed that a department devoted to such investigations is desirable. G.

$$x=-1-2h,$$

$$y=\frac{-2h^2d+3h+1}{b},$$

$$b=\frac{(-2h^2d+3h+1)^2}{4h^2d-4h-1} \quad (b\neq\infty).$$

Eliminating y from the equations of circle and conic gives a biquadratic whose roots are of course the abscissae of intersection points, one of which has been found above. The depressed equation should have a double root to give the desired tangency. The discriminant $\triangle$ of this cubic must vanish, thus giving an equation for the determination of h and thence of a and b in terms of e. For this cubic the coefficients are

$$a_0=d+1,\ 3a_1=h(-2d+2)+3d+7,$$

$$3a_2=\frac{h^3(16d^2)+h^2(-12d^2-56d)+h(8d+44)+3d+11}{N},$$

$$a_3=\frac{h^3(8d^2)+h^2(-4d^2-24d)+h(2d+18)+d+5}{N},$$

in which $N=1-4h^2d+4h$, also

$$\triangle\equiv 4(a_0a_2-a_1^2)(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2=0.$$

This form of $\triangle$ leads to some saving of labor over that of the expanded form.

The following are given as aids in comparison of results:

$$a_0a_2-a_1^2=\frac{4}{9N}[h^2(4d^3-8d^2+4d)+h^3(-8d^2+36d-4)+h^2(2d^2+25d-29)$$
$$+h(d-23)-4]$$

$$-(a_1a_3-a_2^2)=\frac{4}{9N^2}[h^6(16d^4+48d^3)+h^5(-112d^3-192d^2)$$
$$+h^4(4d^3+236d^2+252d)+h^3(20d^2-188d-108)$$
$$+h^2(-2d^2-57d+49)+h(-d+35)+4]$$

$$a_0a_3-a_1a_2=\frac{8}{9N}[h^4(4d^3-4d^2)+h^3(-16d^2+14d)+h^2(2d^2+31d-11)$$
$$+h(d-21)-4].$$

The final equation giving h, is

$$16h^8d^4(d-1)^2-16h^7d^3(4d^2-17d+5)-4h^6d^2(4d^3-31d^2+219d-37)$$

$$+4h^5d(10d^3-80d^2+337d-30)+h^4(4d^4-40d^3+682d^2-997d+36)$$

$$-h^3(4d^3-92d^2+679d-285)+h^2(2d^2-147d+241)-h(9d-71)+7=0.$$

When $d=0$, corresponding to the cardioid, only one of the four roots is available, two of the others being excluded since $b\neq 0$ and the fourth one because $b\neq\infty$, already included in the second special case.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

245. Proposed by S. I. JONES, A. B., Gunter Bible College, Gunter, Texas.

The shell of a hollow iron ball is 4 inches thick, and contains $\frac{1}{5}$ of the number of cubic inches in the whole ball. Find the diameter of the ball.

I. Solution by S. A. COREY, Hiteman, Iowa.

Let r be the radius of the ball; $(r-4)$ will then be the radius of the hollow sphere enclosed by the shell. As the volumes of spheres are proportional to the cubes of their radii, the conditions of the problem require that

$r^3-(r-4)^3=\frac{1}{5}r^3$, or $\frac{4}{5}r^3=(r-4)^3$, whence, $r=\dfrac{4}{1-\sqrt[3]{\frac{4}{5}}}=55.79$ inches, nearly.

II. Solution by M. R. BECK, Cleveland High School, Ohio.

Let r be the radius of the ball, then $\frac{4}{3}\pi r^3=\frac{4}{15}\pi r^3+\frac{4}{3}\pi(r-4)^3$..........(1).
From (1) we have $r^3-60r^2+240r-320=0$..........(2).
Substitute $r=x+\dfrac{320}{x}+20$ in (2), $x^3+\dfrac{32,768,000}{x^3}-11520=0$..........(3).
Solving the quadratic (3), $x=\sqrt[3]{(6400)}$ or $\sqrt[3]{(5120)}$.
Either root makes $r=55.8016$, and the diameter is 111.6032 inches.

Also solved by P. S. Berg, G. W. Greenwood, A. H. Holmes, L. E. Newcomb, D. B. Northrup, J. Scheffer, J. E. Sanders, and G. B. M. Zerr.

AVERAGE AND PROBABILITY.

169. Proposed by HENRY HEATON, Atlantic, Iowa.

*What is the average length of all straight lines that can be drawn within a given square in every possible direction and every possible length from every point of the square; if all the lines are equally distributed about the starting point and equally distributed as to length.

*The problem as restated above is somewhat different from the one solved in our columns last month. As the above conveys the original meaning of the Proposer it is published as a third solution.

III. Solution by the PROPOSER.

Let $ABCD$ be the given square whose side$=a$, and MN one of the straight lines within the square. Draw AE equal and parallel to MN, EF parallel to AB cutting BC in F, and EG parallel to AD cutting CD in G. Put $MN=AE=x$ and $\angle EAD=\theta$. If x and θ were fixed the number of lines of the length a would equal the number of points in the rectangle $EFCG$ whose area is $a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta$. Then if all lines are supposed to be equally distributed about the point M the required average is

$$A_1=\frac{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]xdx}{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]dx}$$

$$=\frac{\frac{a^4}{12}\int_0^{\frac{1}{4}\pi}(2-\tan\theta)\sec^2\theta\, d\theta}{\frac{a^3}{6}\int_0^{\frac{1}{4}\pi}(3-\tan\theta)\sec\theta\, d\theta}=\frac{3a}{4[\log(1+\sqrt{2})^3+1-\sqrt{2}]}.$$

If the lines are supposed to be so distributed as to join every possible pair of points in the square, the required average is

$$A_2=\frac{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]x^2dx}{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]xdx}$$

$$=\frac{\frac{a^5}{60}\int_0^{\frac{1}{4}\pi}(5-3\tan\theta)\sec^3\theta d\theta}{\frac{a^4}{12}\int_0^{\frac{1}{4}\pi}(2-\tan\theta)\sec^2\theta d\theta}=\frac{a}{15}[2+\sqrt{2}+5\log(1+\sqrt{2})].$$

172. Proposed by J. EDWARD SANDERS.

What is the average length of all straight lines that can be drawn within a given triangle?

*Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let a, b, c be the sides; A, B, C the angles of the triangle; u, v variable lengths from A on b, c; w, x variable lengths from B on a, c; y, z variable lengths from C on a, b. Also let $l_1=\sqrt{(u^2+v^2-2uv\cos A)}$, $l_2=\sqrt{(w^2+x^2-2wx\cos B)}$, $l_3=\sqrt{(y^2+z^2-2yz\cos C)}$, M=average length. Then

*Another solution of this problem will be published next month.

$$M=\frac{\int_0^b\int_0^c\int_0^{l_1} l\,du\,dv\,dl+\int_0^a\int_0^c\int_0^{l_2} l\,dw\,dx\,dl+\int_0^a\int_0^b\int_0^{l_3} l\,dy\,dz\,dl}{\int_0^b\int_0^c\int_0^{l_1} du\,dv\,dl+\int_0^a\int_0^c\int_0^{l_2} dw\,dx\,dl+\int_0^a\int_0^b\int_0^{l_3} dy\,dz\,dl}$$

$$=N/D.$$

$$\int_0^b\int_0^c\int_0^{l_1} l\,du\,dv\,dl=\tfrac{1}{2}\int_0^b\int_0^c(n^2+v^2-2uv\cos A)du\,dv$$

$$=\tfrac{1}{6}\int_0^b(3u^2c+c^3-3uc^2\cos A)du=\frac{bc}{12}(2b^2+2c^2-3bc\cos A).$$

$$\therefore N=\frac{2abc(a+b+c)+a^2b^2\cos C+a^2c^2\cos B+b^2c^2\cos A}{12}.$$

This follows, because $\frac{bc}{12}(2b^2+2c^2-3bc\cos A)=\frac{bc}{12}(2a^2+bc\cos A)$.

$$\int_0^b\int_0^c\int_0^{l_1} du\,dv\,dl=\int_0^k\int_0^c\sqrt{(u^2+v^2-2uv\cos A)}du\,dv$$

$$=\tfrac{1}{2}\int_0^b\Big[u^2\cos A+(c-u\cos A)\sqrt{(u^2+c^2-2uc\cos A)}$$

$$+u^2\sin^2 A\log\left(\frac{c-u\cos A+\sqrt{(u^2+c^2-2uc\cos A)}}{n(1-\cos A)}\right)\Big]du$$

$$=\tfrac{1}{6}(b^3+c^3-a^3)\cos A+\tfrac{1}{3}abc\sin^2 A+\tfrac{1}{6}b^3\sin^2 A\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B)$$

$$+\tfrac{1}{6}c^3\sin^2 A\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}C).$$

$$\therefore D=\tfrac{1}{6}[(b^3+c^3-a^3)\cos A+(a^3+c^3-b^3)\cos B+(a^3+b^3-c^3)\cos C+2abc(\sin^2 A$$

$$+\sin^2 B+\sin^2 C)+b^3\sin^2 A\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B)+a^3\sin^2 B\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B)$$

$$+(c^3\sin^2 A+a^3\sin^2 C)\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}C)$$

$$+(b^3\sin^2 C+c^3\sin^2 B)\log(\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C)].$$

If the line is terminated by two of the sides

$$M_1=\frac{\int_0^b\int_0^c l_1\,du\,dv+\int_0^a\int_0^c l_2\,dw\,dx+\int_0^a\int_0^b l_3\,dy\,dz}{\int_0^b\int_0^c du\,dv+\int_0^a\int_0^c dw\,dx+\int_0^a\int_0^b dy\,dz}=\frac{D}{bc+ac+ab}.$$

Corollary. If $a=b=c$, $N=\frac{5}{8}a^4$, $D=a^3(1+\log 3)$.

$$\therefore M=\frac{5a}{8(1+\log 3)}.\quad M_1=\tfrac{1}{3}a(1+\log 3).$$

CALCULUS.

207. Proposed by F. P. MATZ, Sc. D., Ph. D.

If K represents the complete elliptic integral of the first kind, prove that

$$\int_0^1 \frac{Kd\kappa}{1+\kappa}=\tfrac{1}{4}\pi^2.$$

Solution by S. A. COREY, Hiteman, Iowa.

As $K=\int_0^1 \frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}$, the definite integral may be written

$$\int_0^1\int_0^1 \frac{dx\, d\kappa}{(1+\kappa)\sqrt{[(1-x^2)(1-\kappa^2x^2)]}},$$

$$\text{or} \int_0^1 \frac{1}{(1-x^2)^{\frac{1}{2}}}\left[\int_0^1 \frac{d\kappa}{(1+\kappa)(1-\kappa^2x^2)^{\frac{1}{2}}}\right]dx \ldots\ldots\ldots (1).$$

If $r=\kappa+1$, $\int_0^1 \frac{d\kappa}{(1+\kappa)\sqrt{(1-\kappa^2x^2)}}=\int_1^2 \frac{dr}{r\sqrt{X}}=$

$$-\left[\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X}+\sqrt{a}}{x}+\frac{b}{2\sqrt{a}}\right)\right]_1^2=-\frac{1}{\sqrt{a}}\log\left(\frac{1}{1+\sqrt{a}}\right)=\frac{1}{\sqrt{a}}\log(1+\sqrt{a}),$$

in which $a=(1+x^2)$, $b=2x^2$, and $X=(1-x^2)+2x^2r-x^2r^2$.

The definite integral (1), after substituting, becomes,

$$\int_0^1 \frac{\log[1+\sqrt{(1-x^2)}]}{(1-x^2)}dx, \text{ or, if } x=\frac{1-z^2}{1+z^2},\quad \int_0^1 \frac{1}{z}.\log\left(\frac{(1+z)^2}{1+z^2}\right)dz$$

$$=2\int_0^1 \frac{\log(1+z)}{z}dz-\int_0^1 \frac{\log(1+z^2)}{z}dz=2.\frac{\pi^2}{6}-\frac{\pi^2}{2.6}=\frac{\pi^4}{4}.$$

208. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0.$$

Solution by A. H. HOLMES, Brunswick, Me.

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0=\frac{d}{dx}\left[(a^2+x^2)\frac{dy}{dx}\right].$$

$$\therefore (a^2+x^2)\frac{dy}{dx}=C;\ dy=\frac{Cdx}{a^2+x^2};\ y=\frac{C}{a}\tan^{-1}\frac{x}{a}+C'.$$

Also solved by S. A. Corey, M. E. Graber, G.W. Greenwood, J. O. Mahoney, E. L. Rich, J. Scheffer, J. E. Sanders, and G. B. M. Zerr.

GEOMETRY.

268. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find, without the aid of trigonometry, the side of an inscribed regular polygon of $2n$ sides, if the side of an inscribed regular polygon of n sides is 16 feet. [Wentworth's *Plane Geometry*, Revised Edition, problem 512, page 244.]

Solution by M. R. BECK, Cleveland, Ohio.

Let AB be the chord=16. From the center O draw the radius $r=OC$, perpendicular to AB at D. Draw the chord BC=the side of an inscribed regular polygon of $2n$ sides.

In $\triangle OBD$, $r^2=(r-DC)^2+64$, and therefore $DC=r-\sqrt{(r^2-64)}$. Also in $\triangle CDB$, $BC^2=DC^2+64$. Substituting, $BC=\sqrt{\{2r[r-\sqrt{(r^2-64)}]\}}$.

Also solved by P. S. Berg.

269. Proposed by J. SCHEFFER, A. M.

Find the area of a segment, if the chord of the segment is 10 feet, and the radius of the circle is 16 feet.

Solution by P. S. BERG, Larimore, N. D.

Since $16-\sqrt{(16^2-5^2)}=.8014$, height of segment, then

$\frac{(.8014)^3}{2\times10}+\frac{2}{3}\times.8014\times10=5.36$ square feet, area of segment.

Also solved by G. W. Greenwood, M. E. Graber, A. H. Holmes, D. B. Northrup, J. Edward Sanders, and G. B. M. Zerr.

270. Proposed by F. R. HONEY, Ph. B., Hartford, Conn.

What portion of the heavens is always invisible to an observer whose latitude is given?

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

That portion of the heavens which is never visible to an observer in a certain latitude is that cut off by the circle of perpetual occultation. Therefore, if ϕ represents the latitude of the observer, the ratio of the ever invisible portion of the celestial vault to the whole vault is manifestly $\sin^2\frac{1}{2}\phi$.

Also solved by A. H. Holmes, and the Proposer.

271. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two equal concentric ellipses have their axes at an angle θ. Find the area of the quadrilateral circumscribing both, in terms of θ and the semi-axes.

Solution by G. W. GREENWOOD, M. A., McKendree College. Lebanon, Ill.

Denote the equation of one ellipse by $x^2/a^2+y^2/b^2=1$. The quadrilateral circumscribing both is evidently a rectangle whose sides are respectively parallel to the common chords. Their equations are, therefore,

$$x\cos\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta=P_1\,;\quad -x\sin\tfrac{1}{2}\theta+y\cos\tfrac{1}{2}\theta=P_2\,;\quad \text{where } P_2^2=a^2\cos^2\tfrac{1}{2}\theta+b^2\sin^2\tfrac{1}{2}\theta\,;$$
$$P_2^2=a^2\sin^2\tfrac{1}{2}\theta+b^2\cos^2\tfrac{1}{2}\theta.$$

The required area $=4P_1P_2=4\sqrt{[(a^2-b^2)\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta+a^2b^2]}$.

Also solved by A. H. Holmes, A. S. Hawkesworth, and G. B. M. Zerr.

272. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A point A revolves with uniform speed in a circle. A point B revolves around A, at a uniform distance from it, with the same angular velocity, but in the opposite direction. Determine the locus of B.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denote the center of the fixed circle by O, and take Ox through A and B at a time when these points are collinear, A lying between O and B. Let A move counter clockwise. Then $x=(a+b)\cos\theta$, $y=(a-b)\sin\theta$.

$\therefore\ \dfrac{x^2}{(a+b)^2}+\dfrac{y^2}{(a-b)^2}=1$, where a, b are the radii of the fixed and moving circles, respectively.

Also solved by A. H. Holmes, A. S. Hawkesworth, D. B. Northrup, G. B. M. Zerr, and Proposer.

GROUP THEORY.

11. Proposed by DR. SAUL EPSTEEN, University of Colorado, Boulder.

Find the six-parameter continuous group which leaves invariant the surface of second order $x_1x_2-x_3x_4=0$.

Solution by the PROPOSER.

To every transformation of this group defined as below, correspond four different sets of values of the parameters a, b, ... In particular to the identical transformation corresponds $a=d=\pm1$, $\alpha=\delta=\pm1$, $b=c=\beta=\gamma=0$.

The group is

$$\begin{aligned}
x'_1&=\alpha(ax_1+bx_4)+\beta(ax_3+bx_2),\\
x'_2&=\gamma(cx_1+dx_4)+\delta(cx_3+dx_2),\\
x'_3&=\gamma(ax_1+bx_4)+\delta(ax_3+bx_2),\\
x'_4&=\alpha(cx_1+dx_4)+\beta(cx_3+dx_2).
\end{aligned}$$

MECHANICS.

181. Proposed by F. ANDEREGG, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

A triangle AOB, of which the sides, OA, AB, and the angle at O are a, b, and α, revolves uniformly about O, so that OA makes the angle nt with the axis of x, and carries a circle of which AB is the diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe an ellipse whose axes are

$$\sqrt{(a^2+b^2+2ab\cos\alpha)}\pm\sqrt{(a^2+b^2-2ab\cos\alpha)}.$$

Solution by F. H. SAFFORD, Ph. D., The Universitv of Pennsylvania.

Let the third side of the the triangle be c and let the median from O be d, then

$$c=\tfrac{1}{2}\sqrt{(a^2+b^2-2ab\cos\alpha)},\quad d=\tfrac{1}{2}\sqrt{(a^2+b^2+2ab\cos\alpha)}.$$

Consider the triangle at the time when the moving point is on the produced median and take the x-axis as coincident with the position of the median at this time. By taking the rotation of the moving point in the direction *opposite* to that of the triangle and noticing that the circle is itself subject to the rotation of the triangle on which it is fixed, the double velocity of the point causes it to have with respect to the axis of x a rotation opposite in direction and equal in amount to that of the triangle. Hence the coördinates of the moving point are

$$x=d\cos\theta+c\cos\theta,\ y=d\sin\theta-c\sin\theta,\ \text{or}\ x=(d+c)\cos\theta,\ y=(d-c)\sin\theta,$$

from which follows the ellipse with axes as stated.

The problem may be executed mechanically as follows. At the point O fasten to the fixed plane a circle of twice the diameter of the carried circle. This circle is to remain fixed and to be connected with the carried circle by an *open* belt.

185. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A perfectly flexible rope whose weight is w per linear unit, and length $2l$, rests in equilibrium on a smooth peg. If now one end be raised a distance a and then released, find the time in which this end will rise to the height x above its original position, and the tension at that instant of the rope at the point where it passes over the peg.

Solution by G. W. GREENWOOD, M. A., McKendree College. Lebanon, Ill.

Denote by z the height of one end above the initial position, and let m be the mass per unit of length. Then

$$2lm\frac{d^2z}{dt^2}=2mgz.\quad\therefore 2\frac{dz}{dt}\frac{d^2z}{dt^2}=\frac{2gz(dz/dt)}{l}.$$

$$\left(\frac{dz}{dt}\right)^2=\frac{g}{l}(z^2-a^2),\text{ since } z=a \text{ when } \frac{dz}{dt}=0.$$

$$\therefore t=\sqrt{\frac{l}{g}}\int\frac{dz}{\sqrt{(z^2-a^2)}}=\sqrt{\frac{l}{g}}\log\frac{z+\sqrt{(z^2-a^2)}}{a},$$

if $t=0$ when $z=a$. Putting $z=x$ we get the required time.

Also we have, [Tension at peg]δt=momentum generated by tension in time $\delta t=mv\delta t.v=mv^2\delta t$.

$\therefore$ Tension at peg$=m\left(\frac{dx}{dt}\right)^2=\frac{mg}{l}(x^2-a^2)$, when $z=x$.

Also solved by S. A. Corey, and G. B. M. Zerr.

MISCELLANEOUS.

152. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A conductor, the equation of the surface of which is

$$\frac{x^2}{25}+\frac{y^2}{16}+\frac{z^2}{9}=1,$$

is charged with 80 units of electricity, what is the density at a point for which $x=3$, $y=3$? If the density of this point be a, what is the whole charge on the ellipsoid? [From Peirce's *Potential Functions*, example 165, p. 388.]

Solution by M. E. GRABER, A. M., Tiffin, Ohio.

The mass of an ellipsoidal shell is $\frac{4}{3}\pi\rho d(abc)=4\pi Mabc$, and $Q=A4\pi\mu abc\rho$. $s=A\mu\theta$ and $\theta=\mu p$ where p is the perpendicular from the origin on the tangent plane. Then the density at any point is $Qp/4\pi abc$. The value of z for a point on the ellipsoid for which $x=3$ and $y=3$, is $\frac{3\sqrt{31}}{20}$ and the equation of the tangent plane is $\frac{3x}{25}+\frac{3y}{16}+\frac{3\sqrt{31}}{9}z=1$. The perpendicular from $(0, 0, 0)$ is $\frac{D}{\sqrt{(A^2+B^2+C^2)}}$ or $4.2+$. Therefore the density at $\left(3, 3, \frac{3\sqrt{21}}{20}\right)=\frac{80.(4.2)}{4\pi(5.4.3)}$ $=.445+$. If the density of this part be a,

$$Q=\frac{4\pi a(5.4.3)}{4.2+}=\frac{240\pi a}{4.2+}.$$

Also solved by G. B. M. Zerr, and the Proposer.

153. Proposed by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

Two men start from Columbus, Ohio, at the same time; one travels east and the other west. They travel at the rate of 4 miles an hour from sunrise to sunset each day until they meet. Where will they meet and what distance will each have traveled?

Solution by A. H. HOLMES, Brunswick, Me.

We assume that Columbus, Ohio, is situated at 40° N. latitude, and 83° W. longitude. $2\pi x$=distance round the earth at 40° latitude. Put $a=\frac{1}{2}$ equatorial diameter of the earth, and $b=\frac{1}{2}$ polar diameter of the earth.

Then $a^2y^2+b^2x^2=a^2b^2$. $y=x\tan 40°$.

$\therefore 3962.8^2(.704088x^2)+3949.57^2x^2=3962.8^2\times 3949.57^2$. $\therefore x=3031.48$.

Then A going east will have travelled when he meets B, $\frac{8\pi x-384}{16\pi x}\times 2\pi x=\pi x-48$; and B, $\pi x+48$ miles.

$\therefore$ A travelled 9475.6899+ miles, B travelled 9571.6899+ miles. A will have passed 179° 5′ of longitude.*

Hence they will meet at a point in the Eastern Desert of the Chinese Empire situated at 40° N. latitude, and 96° 5′ E. longitude.

Also solved by L. E. Newcomb, J. Scheffer, and G. B. M. Zerr.

154. Proposed by D. BIDDLE (Unsolved problem in the Educational Times, London).

Prove that the proper angle at which to cross a street when a person wishes to continue his course on the other side, and the roadway is n times as muddy as the pavement, is that of which the sine is $(n^2-1)/(n^2+1)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Since the roadway is n times as muddy as the pavement, the velocity in the roadway is $1/n$th the velocity on the pavement.

Let A be the position of the person, B the point directly opposite him on the other side of the roadway, C the point where he reaches the other side, $AB=a$, $\angle CAB=\frac{1}{2}\pi-\theta$. Also let v be his velocity on the pavement. Then to get Mr. Biddle's result, the time to go over AB plus the time to go over BC, or

$$\frac{n.AB}{v}+\frac{BC}{v}=\frac{n.AC}{v}. \quad \therefore \frac{na}{v}+\frac{a\cot\theta}{v}=\frac{na\operatorname{cosec}\theta}{v}.$$

$$\therefore n\sin\theta+\cos\theta=n. \quad \therefore \sin\theta=\frac{n^2-1}{n^2+1}.$$

I prefer the following result: Let D be the person's goal, $BD=b$, $AC=a\operatorname{cosec}\theta$, $BC=a\cot\theta$.

$$\text{Then } \frac{na\operatorname{cosec}\theta}{v}+\frac{b-a\cot\theta}{v}=\text{minimum}.$$

$$\therefore n\operatorname{cosec}\theta\cot\theta=\operatorname{cosec}^2\theta \text{ or } \cos\theta=1/n. \quad \therefore \sin\theta=\sqrt{(n^2-1)}/n.$$

Also solved by A. H. Holmes, and J. Scheffer.

*Mr. Newcomb in his solution used more exact assumptions in regard to the latitude of Columbus, and tabulated the data showing the progress of A snd B on successive days. The result of his long solution is 179° 33′ 13.1″ East of Columbus. G.

PROBLEMS FOR SOLUTION.

ALGEBRA.

250. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Factor $a^2b^2(x^2+y^2)(a^2y^2+b^2x^2-a^2b^2)=(a^4y^2+b^4x^2)[\sqrt{(a^2y^2+b^2x^2)}+ab]^2$.

251. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}+\dfrac{1}{3(n+3)}+$ etc., $=$

$$\frac{1}{n^2}+\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\cdots\cdots+\frac{1}{l(n-l)}\right],$$

l being equal to $(n-1)$, n being any positive integer greater than one.

252. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x-y=\frac{1}{3}\pi$; (2) $\sin x=\cos^3 y$.

AVERAGE AND PROBABILITY.

174. Proposed by HENRY HEATON, Atlantic, Iowa.

Chords are drawn through every point of the surface of a given circle in every possible direction. What is their average length?

175. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a line l is divided into three parts by two points taken at random on it, what is the mean value of the triangle whose sides are equal to the three parts? (Only those cases are to be considered in which the the three parts will form a triangle.)

CALCULUS.

212.* Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Show that any root of the equation $y^5-5y=4x$ satisfies the differential equation $\dfrac{d^4y}{dx^4}=(\frac{4}{5})^4\,x^{-3}\dfrac{d^4(x^{-\frac{1}{5}}y)}{d(x^{-\frac{4}{5}})^4}$. Generalize the problem.

213. Proposed by EDWIN L. RICH, Schenectady, N. Y.

Let $f(x)$ be any function of x, and $f'(x)$ its derivative. If $u=[f'(x)]^{-\frac{1}{2}}$, $v=f(x)[f'(x)]^{-\frac{1}{2}}$, then $\dfrac{1}{u}\dfrac{d^2u}{dx^2}-\dfrac{1}{v}\dfrac{d^2v}{dx^2}=0$.

*The problem is due to Heymann.

214. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Prove that $\pi^2=6.\frac{2^2}{2^2-1}.\frac{3^2}{3^2-1}.\frac{5^2}{5^2-1}$......... where the squared numbers in the numerator are the natural *primes* in order.

GEOMETRY.

277. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It is tacitly assumed in elementary geometry that as the number of sides of a regular polygon inscribed in a circle is increased, *in any manner*, that its perimeter has a fixed limit. Beginning with a square and then continually doubling the number of sides we get for the perimeter $2^{n+2}\sqrt{[2-E^n(0)]}$, where $E(x)\equiv\sqrt{(2+x)}$. Beginning with a hexagon we get $2^{m+1}3\sqrt{[2-E^m(1)]}$. The definition of the length of a circle assumes that these expressions have the same limit as $n\doteq\infty$ and $m\doteq\infty$. Prove it.

278. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

AF, MN are parallel lines indefinitely extended toward FN; at right angles to AF, MN is AM of length 22; upon the base AB, which is in line with AM, is the triangle ABC whose sides are $AB=21$, $BC=10$, $AC=17$; find the sides of the maximum similar triangle with base extending from B to some point in AF, the vertex in line with MN.

279. Proposed by C. C. WENTWORTH, C. E., Roanoke, Va.

To construct geometrically the maximum equilateral triangle circumscribed about a given triangle.

GROUP THEORY.

12. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics, The University of Pennsylvania.

*Given $U_1=\alpha'$, $V_1=\beta'$, and the recursion formulae $U_y=\alpha' V_{y-1}+\alpha'' U_{y-1}$, $V_y=\beta' V_{y-1}+\beta'' U_{y-1}$. Find expressions for U_y, V_y in terms of the coefficients α', α'', β', β''.

13. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The order of the linear homogeneous group in n letters is $(p^n-1)(p^n-p)$ ....(p^n-p^{n-1}). Two proofs are given in Burnside's *Finite Groups*. Give other proofs.

*The problem is of frequent occurrence in abstract group construction.

MISCELLANEOUS.

155. Proposed by A. H. HOLMES, Brunswick, Maine.

There are two vessels, one containing a gallons of alcohol, the other b gallons of water. Suppose that c gallons are simultaneously taken from each and poured into the other, how many times must this be done so that there will be the same proportion of alcohol to water in each vessel?

156. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

There exist no multiply perfect odd numbers of multiplicity n containing only n distinct primes.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Average and Probability, 164 Proposed by J. O. MAHONEY, B.E., M.Sc., Central High School, Dallas, Tex.

If m is prime, and the numbers 0, 1, 2, ..., m^2-1 are placed at random in the form of a square, the probability that the square is hyper-magic is $(m-1)m/(m^2-2)!$

Algebra, 179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

NOTES AND NEWS.

Mr. B. M. Rastall is assistant in mathematics at the University of Wisconsin.

Mr. E. A. Moritz, Mr. R. S. Peatter, and Mr. E. R. Smith have been appointed instructors in mathematics at the University of Wisconsin.

The University of North Carolina has granted a years leave of absence to Mr. M. H. Stacy, who will pursue a graduate course in mathematics at Cornell University.

The American Association for the Advancement of Science held its annual meeting during the holidays at New Orleans, La. Section A: Mathematics and Astronomy, was presided over by Dr. W. S. Eichelberger, of the United States Naval Observatory at Washington, D. C. Professor L. G. Weld, of the University of Iowa, is secretary of the Section.

The graduate department of the University of Cincinnati has been reorganized with the title of graduate school. Its faculty consists of the heads of departments of the University.

The Mathematical Club of the University of Washington was organized this year, with Professor Moritz, president, F. M. Morrison, secretary. The meetings of the Club are monthly and are devoted to reports on original work, and the reviews of current mathematical literature.

A comparison of the membership of the world's mathematical societies of national character places Germany first and the United States second. The Deutsche Mathematiker Vereinigung has 666 members, while the membership of the American Mathematical Society is slightly over 500. The London Mathematical Society has between 290 and 300 members.

John Henry Serviss, civil engineer and surveyor, died at his residence in Closter, Bergen County, New Jersey, on August 26th. Mr. Serviss was born in the town of Glen, Montgomery County, New York, August 17th, 1837, and was graduated from Union College in 1863. Since 1868, he has practiced his profession as civil engineer throughout northern New Jersey with distinguished ability.

During the past year there was created at the University of Washington a new assistant professorship of mathematics which was filled by the appointment of Professor F. M. Morrison. Over $800 worth of new books have been added to the mathematical library during the past year and $200 worth of models. This progress in the department of mathematics is in keeping with the general growth of that university.

The following have been elected members of the American Mathematical Society: Professor O. P. Akers, Allegheny College; Dr. R. B. Allen, Clark University; Professor Ernesto Cesàro, University of Naples; Lieutenant Colonel A. J. C. Cunningham, London, Eng.; Miss M. E. Decherd, University of Texas; Mr. W. W. Hart, Shortridge High School, Indianapolis, Ind.; Mr. H. N. Olsen, Bethany College; Mr. F. H. Smith, Southwestern Christian College.

The Missouri Society of Teachers of Mathematics held its regular Winter meeting at Jefferson City on December 27 and 28. The following papers were read: Maxima and Minima, by George R. Dean, Missouri School of Mines; Laboratory Methods in Algebra Teaching, by Oliver E. Glenn, Drury College; The Treatmeht of Limits in Elementary Geometry, by Albert M. Wilson, McKinley High School, St. Louis; Some Problems in Arithmetic in the Grades, by Thos. P. Jandon, Kansas City. The Thursday afternoon program was devoted to a round table discussion, "what should be taught in arithmetic and what omitted."

We learn from *Science* that the annual meeting of the Association of Teachers of Mathematics in the Middle States and Maryland was held on Saturday, December 2, in affiliation with the Association of Colleges and Preparatory Schools of the Middle States and Maryland. The following papers were presented:

Professor H. S. White, Vassar College, How Should the College Teach Analytic Geometry. Mr. H. R. Higly, Pennsylvania College: Suggestions for the First Twelve Lessons in Demonstrative Geometry. Dr. John S. French, Jacob Tome Institute, Port Deposit: Some Essentials of the Successful Mathematics Teacher. Dr. H. A. Converse, Baltimore Polytechnic Institute: The Teaching of Geometry.

The association was disappointed at not being able to listen to a paper on The Teaching of Pure and Applied Mathematics, which the program announced was to be read by President R. S. Woodward, of Carnegie Institute, Washington.

The following officers were elected for the coming year: President, Professor Edwin S. Crawley, University of Pennsylvania. Vice President, Dr. John S. French, Jacob Tome Institute, Port Deposit, Md. Secretary and Treasurer, Dr. J. T. Rorer, Central High School, Philadelphia, Pa. Members of the Council, Professor W. H. Metzler, Syracuse University; Miss L. G. Simons, New York City Normal College; Dr. J. L. Patterson, Chestnut Hill Academy, Philadelphia, Pa.; Professor W. H. Maltbie, Woman's College of Baltimore.

At the meeting the following resolution was adopted:

Resolved, That this association approve of the organization of a national federation of existing associations of teachers of mathematics in which each association shall preserve its own organization and individuality and which shall have among its objects the joint support of a publication. In the federation should be included only societies representing territory as extensive at least as one State.

The Association of Ohio Teachers of Mathematics and Science holds its third annual meeting in Townsend Hall, Ohio State University, Columbus, on December 28. The following papers will be read before the mathematical section: Do the mathematical courses in literary colleges property fit for the mathematics of engineering?—Christian Hornung, Tiffin. Recent contributions marking a real advance in the theory of mathematics teaching.—Anna H. Palmie, Cleveland. Sir Isaac Newton—an estimate.—C. L. Arnold, Columbus. A contribution from non-Euclidean geometry to school spherics.—George Bruce Halsted. Symbolism in mathematics.—Harriet E. Glazier, Oxford. The character of mathematics teaching in the high school.—W. T. Heilman, Columbus. Report of the committee on "A straight line is the shortest distance or path between two points.—George Bruce Halsted, Gambier. Preliminary report of the committee on a syllabus of the fundamental propositions of elementary geometry proposed for consideration by the Association of Mathematical Teachers in New England. —George Bruce Halsted, Gambier. The influence of college entrance certificates on the teaching of mathematics in the high school.—Harry E. Giles, Kenton.

BOOKS AND PERIODICALS.

Theory of Functions of Real Variables. By James Pierpont, Professor of Mathematics in Yale University. Vol. I. 8vo. Cloth, xii+560 pages. Price, $4.50. Boston: Ginn & Co.

This work, which the author says is based upon a course of lectures given by him at Yale University, is designed to present the more important results in the theory of functions of real variables. Vol. I is devoted to the foundations of the Differential and Integral Calculus. It is divided into sixteen chapters. In Chapter I, rational numbers, the laws governing their generation, and some of their properties are discussed; Chapter II, deals in the same way with irrational numbers; Chapter III, with exponentials and logarithms; Chapter IV, with elementary functions; Chapter V, with first notions of point aggregates; Chapter VI, with limits of functions; Chapter VII, with continuity and discontinuity of functions; Chapter VIII, differentiation; Chapter IX, with implicit functions; Chapter X, with indeterminate forms; Chapter XI, with maxima and minima; Chapter XII, with integration; Chapter XIII, with proper integrals; Chapter IV, improper integrals, integrand infinite; Chapter XV, improper integrals, interval of integration infinite; and Chapter XVI, multiple proper integrals.

The theorems are first explicitly stated and then proved. Many examples of incorrect forms of reasoning currently found in standard works on the calculus are given to emphasize the necessity of a more critical study of fundamentals and at the same time "to stimulate the critical sense of the student." While that part of the work developing the theory of irrationals as introduced by Cantor and Dedekind, the theory of point aggregates or masses of points, and the theory of discontinuous functions are not as fully treated as those familiar with the profound researches of Cantor, Weierstrass, Neumann, and others might wish, yet the author for writing and the publishers for publishing it, have placed the American mathematical world under obligations to them. It is extremely gratifying to see an American publishing company taking upon itself the responsibility of publishing such works as this and Goursat's *Cours d'Analyses*, with no possibility of great financial returns, but solely with a view of encouraging the development of mathematics in America. They should receive, therefore, every possible encouragement for the interest they are thus manifesting in disseminating by such publications, the refined views of modern mathematicians.

B. F. F.

A Course in Mathematical Analysis. By Édvard Goursat, Professeur à la Faculté des Sciences de Paris. Authorized translation by E. R. Hedrick, Professor of Mathematics in the University of Missouri. Vol. I. 8vo. Cloth, viii+548 pages. Illustrated with 52 diagrams. Price, $4.00. Boston: Ginn & Co.

The French edition of this valuable work appeared in 1902, and a critical review of it, by Professor W. F. Osgood, of Harvard University, appeared in the *Bulletin of the American Mathematical Society*, Vol. IX, No. 10, 1903. At Professor Osgood's suggestion, the translation of the work into English was undertaken.

In this work, the fundamental principles of the calculus are established with great rigor and thoroughness. It begins with the foundations of the calculus, assuming, however, that the student has already had an elementary course in this subject. In the first chapter, precise definitions of *limit, function, derivative,* etc., are given. Chapter II is an exposition of the modern treatment of implicit functions, functional relations, and transformations. In Chapter III, Taylor's Theorem and its applications are presented in an admirable manner. Also the subject of maxima and minima is treated with equal rigor and elegance. The ambiguous case is treated with considerable fullness. The word *extremum* is used as a generic term for maximum and minimum. Chapters IV, V, VI, and VII deal

with definite and indefinite integrals from the modern point of view. The definite integral is defined as sum and the fundamental properties and formulas are established from it. In Chapter VIII, infinite series, including series of imaginary terms, and the theory of uniform convergence are discussed. Chapter IX supplements Chapter VIII with a treatment of power series and trigonometric series, including Fourier's Series. Chapters X, XI, and XII deal with curves and surfaces.

The work is one that is sure to appeal strongly to those teachers of mathematics who wish to put into the hands of their students a book eminently scholarly and accurate in treatment and embracing the most modern materials and applications. An equal service would be rendered to the American mathematical public, were some one to put into good English with the necessary annotations and proofs of many of the unproved theorems of Dini's *Theory of Functions*, and other epoch-making works on mathematics which are generally unknown to most teachers of mathematics. B. F. F.

The current number of the *Proceedings of the London Mathematical Society* contains the following papers:

The Intersection of Two Conic Sections (continued), by Mr. J. A. H. Johnston; On the Projection of Two Triangles on the Same Triangle, by Professor M. J. M. Hill, Dr. L. N. G. Filon, and Mr. H. W. Chapman; On the Condition of Reducibility of any Group of Linear Substitutions, by Professor W. Burnside; On a Class of Analytic Functions, by Mr. G. H. Hardy; Linear Content of a Plane Set of Points, by Dr. W. H. Young.

ERRATA.

On page 207, line 5 from bottom, for $\dfrac{d^2\,\theta'}{dt'}$ read $\dfrac{d^2\,\theta'}{dt'^2}$.

On page 208, line 6 trom bottom, for $\dfrac{d^2\theta''}{d\,t''_2}$ read $\dfrac{d^2\theta''}{dt''^2}$.

On page 211, line 22 should read, $x^5+qx^2+px+\frac{1}{5}\left[\dfrac{q^2}{p}-\dfrac{p^3}{5q}\right]=0$.